Eigenspaces of Graphs and their Utility

Garrett J. Kepler
Joint Mathematics Meeting
January 10th, 2025

Background

Definition

The *Laplacian* matrix $L = [\ell_{ij}] \in M_n$ is defined such that

$$\ell_{ij} = \begin{cases} d_i & \text{if } i = j \\ -1 & \text{if } i \text{ adjacent to } j \\ 0 & \text{otherwise} \end{cases}$$

Definition

Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of L. We call the second smallest eigenvalue of L, λ_2, and its corresponding eigenvector the *Fiedler value* and *Fiedler vector* respectively.

Combinatorial View of Fiedler Value

Theorem

Since the constant vector $\mathbb{1}$ is an eigenvector of L corresponding to 0, by Rayleigh-Ritz, the Fiedler value of L corresponds to the following minimum:

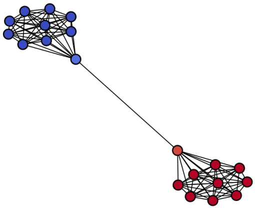
$$\lambda_2 = \min_{x \perp \mathbb{1}} \frac{x^T L x}{x^T x} = \min_{x \perp \mathbb{1}} \frac{\sum_{i \sim j} (x(i) - x(j))^2}{\sum_i x(i)^2}$$

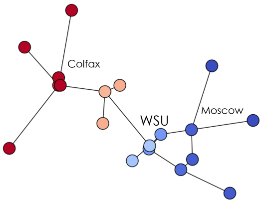
Fiedler Partitioning Theorem

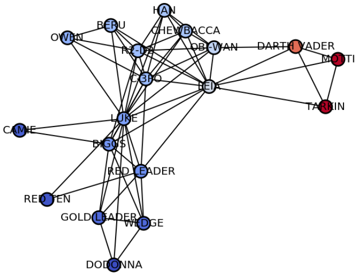
Theorem

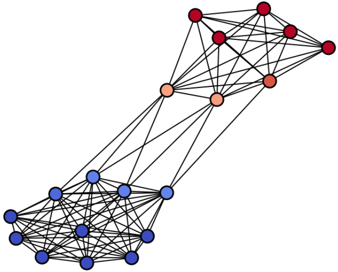
Let $G = (V, E)$ be a connected graph and let x be its corresponding Fiedler vector. Then the subgraphs induced by the vertex sets $V_1 = \{i : x_i > 0\}$ and $V_2 = V \setminus V_1$ are connected.

Informally: we can partition G in to two connected components using the entries of the Fiedler vector.




Colfax
WSU
Moscow

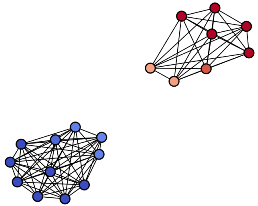

HAN
BERU
CHEWBACCA
OWEN
R2-D2
OBI-WAN
DARTH VADER
MOTTI
C-3PO
LEIA
TARKIN
LUKE
CAMIE
BIGGS
RED LEADER
RED TEN
GOLD LEADER
WEDGE
DODONNA

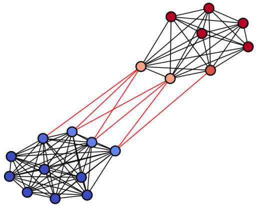


Reconstruction Question

Question

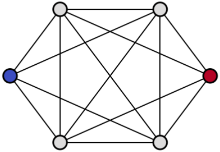
Given the two connected components of a Fiedler-partitioned graph, can we determine what set of edges are missing using the entries of the Fiedler vector?



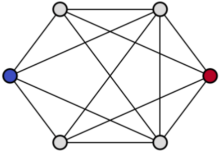


Answer

Not always.



$$x = [1, 0, 0, -1, 0, 0]^T$$
$$\lambda = 4$$



$$x = [1, 0, 0, -1, 0, 0]^T$$
$$\lambda = 4$$

Theorem

Let x be an eigenvector of L corresponding to λ. If $x(i) = x(j)$ for some entries of x, the inclusion or removal of edge (i, j) maintains x as an eigenvector and λ as its corresponding eigenvalue.

Main Characterization Questions

Question

- Can we find graphs with Fiedler vectors that reveal as little information as possible about the cut edges?
- Can we find graphs with Fiedler vectors that reveal as much as information possible?

Method

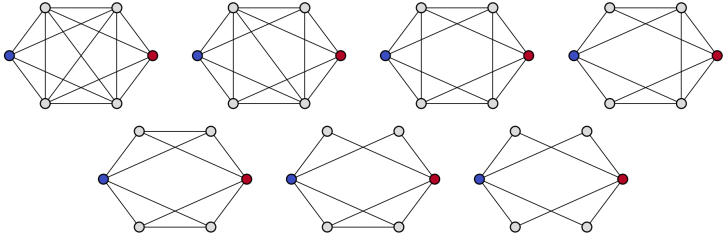
An Approach: Construct graph around vector

Pick x such that

- $x \perp \mathbb{1}$
- x has as many/as little repeated entries as possible
- find graph that has x as Fiedler vector

An Example

- Consider the vector from before: $x = \left[1, 0, 0, -1, 0, 0\right]^T$
- What graphs can we make with x as its Fiedler vector?



1. Preliminaries
2. Characterization
3. **Classification**
4. Application
5. Future Work

Main Classification Question

Question

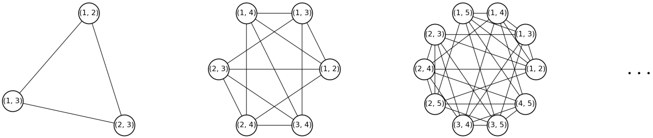
Given a graph, can we determine how much information about the missing edges can be revealed by its Fiedler vector?

Method

An Approach: Construct vector around class of graphs

- Pick a class of graphs
- Characterize their Fiedler vectors
- Observe what eigenvector preserving perturbations are permitted
- Obtain graphs that share that Fiedler vector

An Example



<u>$J(2, n)$</u>:

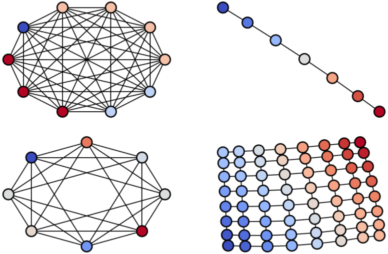
- Constant vector ($\lambda = 0$)
- Assign 1 to node $\{a, b\}$, -1 to node $\{c, d\}$, and 0 elsewhere ($\lambda = n$)
- Assign 1 to pair of nodes $\{a, b\}$ & $\{b, c\}$, -1 to pair of nodes $\{a, d\}$ & $\{c, d\}$, and 0 elsewhere ($\lambda = 2(n - 1)$)

(<u>**Bonus**</u>: eigenvector construction above generalizes to $J(k, n)$ by merely keeping track of intersections!)

Already Characterized Classes

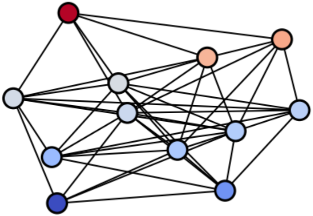
- Complete graphs ✓
- Path graphs ✓
- Grid graphs ✓
- Circulant graphs ✓

⋮



Dimension Reduction

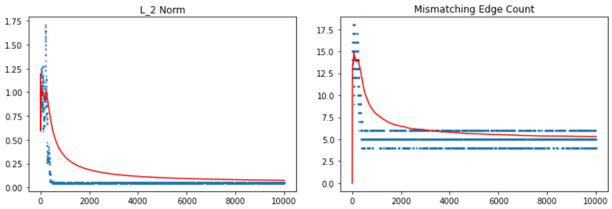
- In the case that we are able to reconstruct some number of missing edges, we are able to reduce the amount of required data from a graph to determine bottlenecking information



$$\longmapsto \begin{bmatrix} 0.642\ldots \\ -0.091\ldots \\ -0.070\ldots \\ 0.312\ldots \\ -0.178\ldots \\ -0.005\ldots \\ 0.040\ldots \\ 0.023\ldots \\ 0.267\ldots \\ -0.312\ldots \\ -0.508\ldots \\ -0.119\ldots \end{bmatrix}$$

MCMC Methods

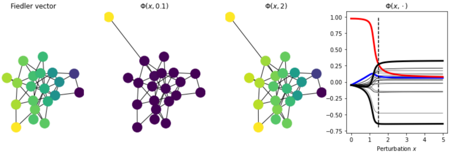
- Some recent work has been done investigating MCMC approximation of graphs via the Fiedler vector
- Classifying the reconstructibility of graphs via the Fiedler vector can aid in the reliability of these methods

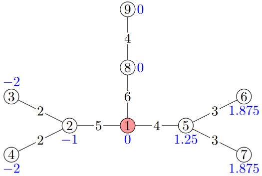


Future Work

- In finding graphs with maximally informative Fiedler vectors, we cleverly picked vectors we knew would be an eigenvector for some graph. What are the constraints for entries that guarantee finding such a graph is possible?
- Extending to k-eigenvector partitioning, what subset of eigenvectors do we need to maximize/minimize the information we can reconstruct via their entries?

Other Interesting Work





- "Perturbation of Fiedler vector: interest for graph measures and shape analysis" by Lefevre, Fraize, & Germanaud
  url: https://arxiv.org/abs/2306.04327
- "Inverse Fiedler vector problem of a graph" by Lin & Shirazi
  url: https://arxiv.org/abs/2410.09736

Thank you!

garrett.kepler@wsu.edu