MCMC: An Introduction

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Coin Flipping

- Let's try flipping some coins!

WLLN

Theorem (Weak Law of Large Numbers)

If $X_1, \ldots, X_n$ are independent and identically distributed random samples, $\bar{X}_n \xrightarrow{P} \mu$ as $n \to \infty$. That is, the sample average converges to the mean of the distribution each X_i follows

Pavel Nekrasov believed independence was necessary for this to hold and also claimed this had more metaphysical implications.

Figure 1: Pavel Nekrasov

Markov proved this is not the case using Markov Chains!

Figure 2: Andery Markov

Markov's Example

He was too young to have been blighted
by the cold world's corrupt finesse;
his soul still blossomed out, and lighted
at a friend's word, a girl's caress.
In heart's affairs, a sweet beginner,
he fed on hope's deceptive dinner;
the world's éclat, its thunder-roll,
still captivated his young soul.
He sweetened up with fancy's icing
the uncertainties within his heart;
for him, the objective on life's chart
was still mysterious and enticing—
something to rack his brains about,
suspecting wonders would come out.

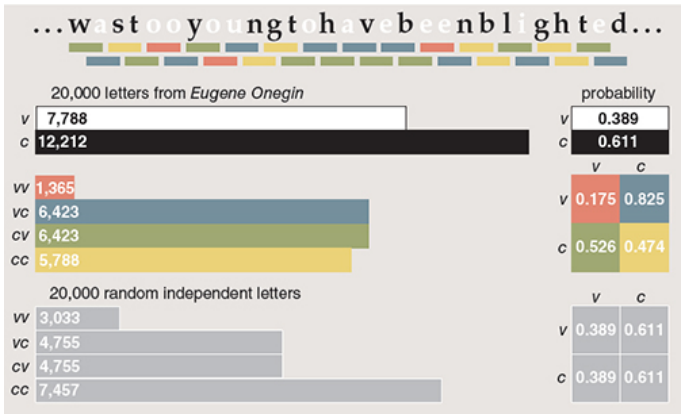

Figure 3: The distribution of consanants and vowels Markov found

Definition

A Markov Chain is a sequence of events whose probability of transitioning to another event is only dependent on the current event.

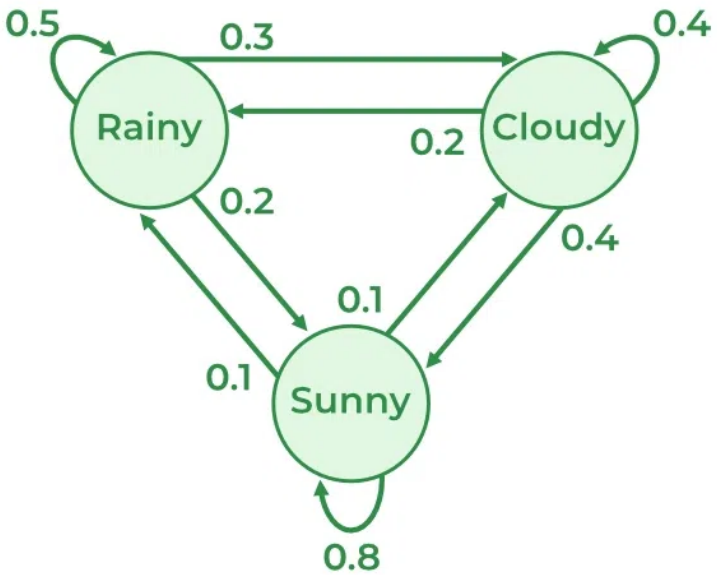

Figure 4: Basic Markov Chain

Matrix View

The matricial view aids in describing Markov Chains using stochastic matrices. Consider the weighted case of our coin flip game:

$$\begin{bmatrix} .75 & .25 \\ .25 & .75 \end{bmatrix}$$

If we wish to assess the states of our heads or tails game at time step k, we merely have to raise this matrix to the power k.

Inventor of Monte Carlo

Over a game of solitaire during a bed-ridden illness, Ulam invented what we call Monte Carlo.

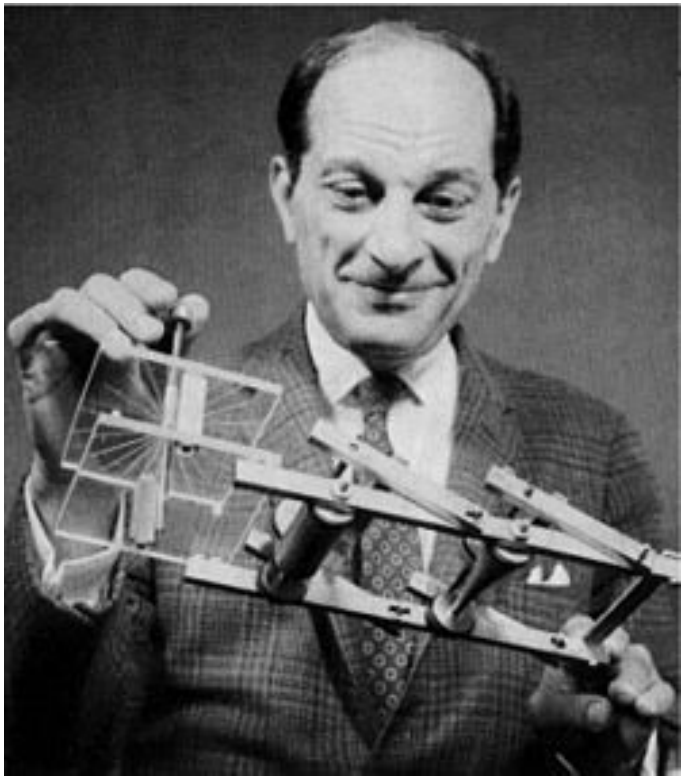

Figure 5: Stanislav Ulam, creator of Monte Carlo methods

Monte Carlo

The Monte Carlo method can be summarized as follows:

- Define a domain
- Generate samples from a distribution over domain
- Apply a function to determine outputs
- Accept sample or reject sample according to your desire

Example

We can use this to approximate π! (Buffon's Needle in Google Collab)

Combine the Two!

The MCMC method combines the two ideas:

- Start with an initial state X_0
- Propose a next step X'
- Compute acceptance probability $\alpha = \min\left(1, \frac{\mathbb{P}[X_0 \to X']}{\mathbb{P}[X' \to X_0]}\right)$
- Sample $U \sim \mathit{Uniform}(0, 1)$
- If $U \leq \alpha$, transition to X'. Else, stay put and try again!

Let's return to Google Collab to give this a try.

Gerrymandering

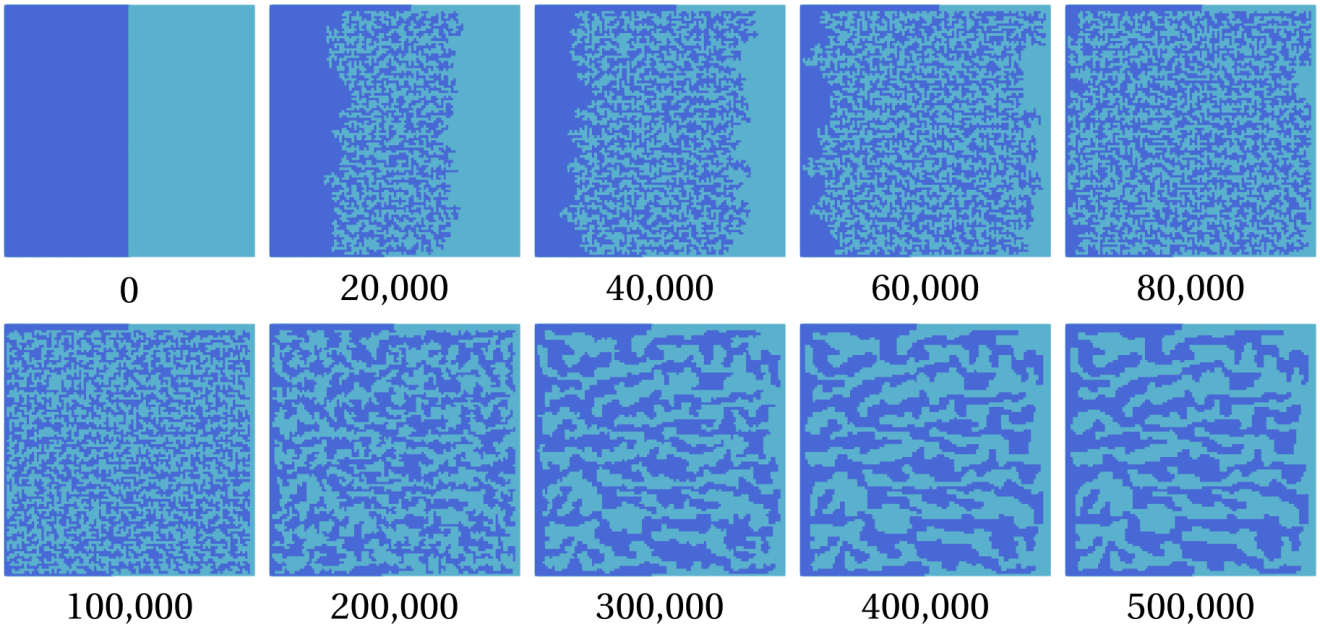

Figure 6: Generating sample of a state (guess which one?)

Gerrymandering

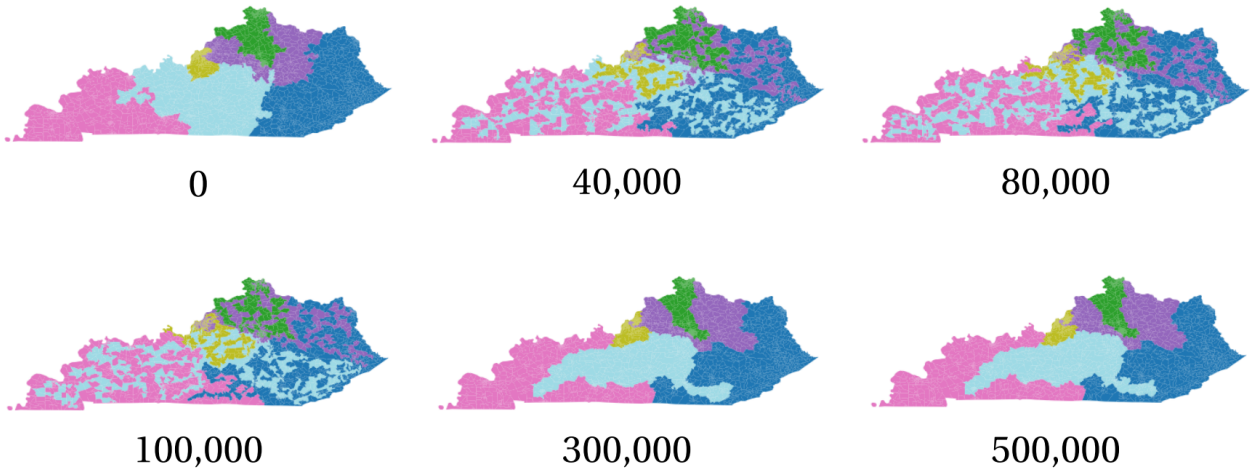

Figure 7: Generating sample of a state (guess which one?)

Gerrymandering

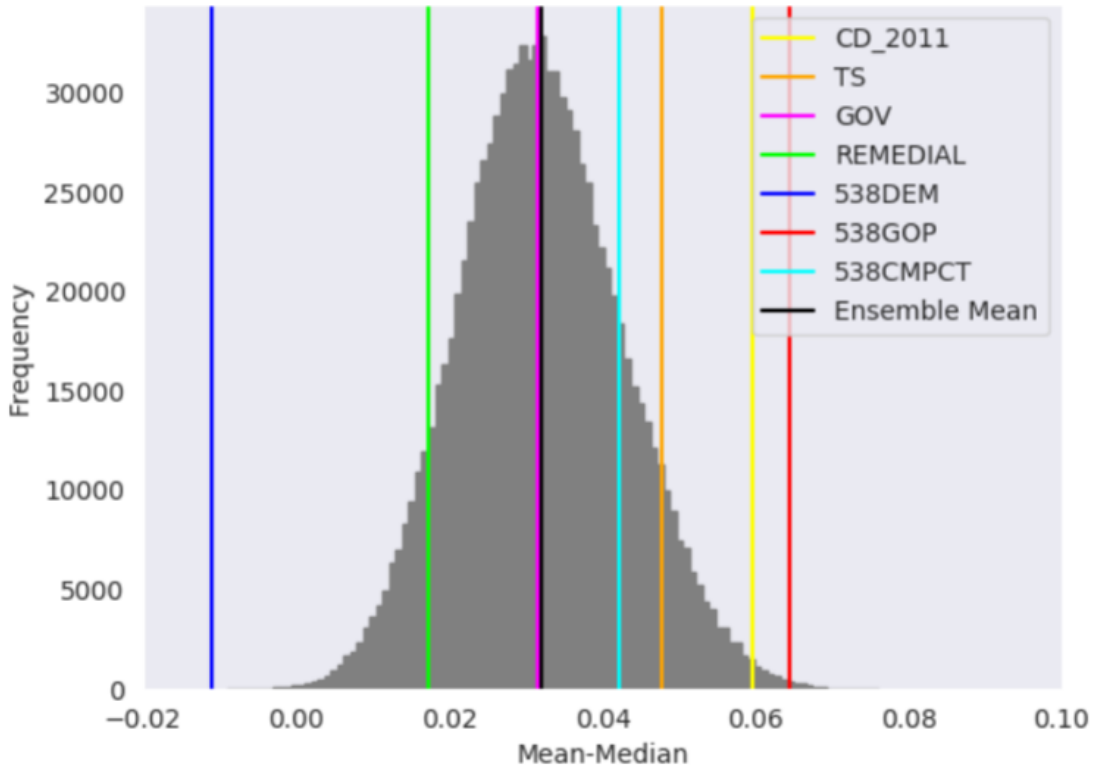

Figure 8: Ensemble approach in Gerrymandering

Cryptography

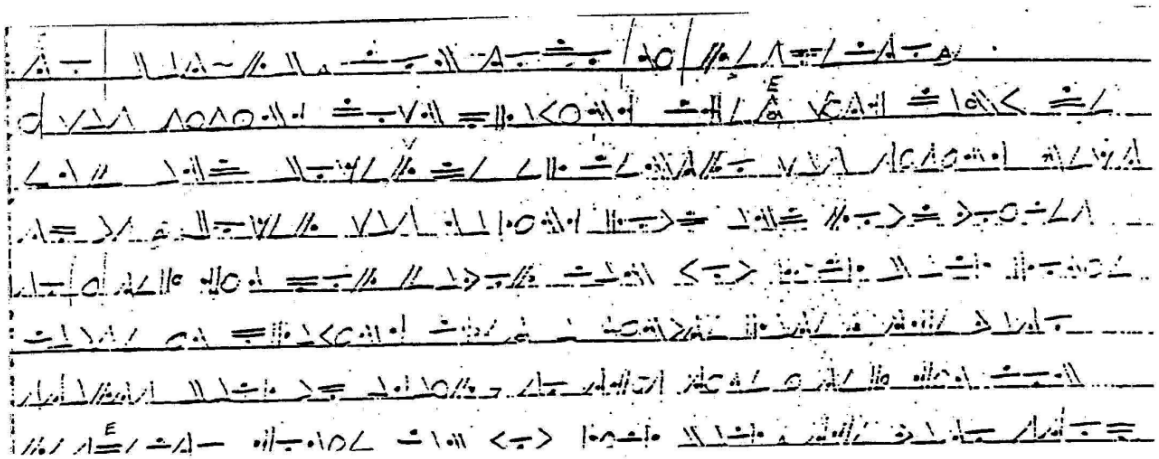

Figure 9: An encrypted message from inside a prison in the Bay Area, California

Cryptography

ENTER HAMLET HAM TO BE OR NOT TO BE THAT IS THE QUESTION WHETHER TIS
NOBLER IN THE MIND TO SUFFER THE SLINGS AND ARROWS OF OUTRAGEOUS
FORTUNE OR TO TAKE ARMS AGAINST A SEA OF TROUBLES AND BY OPPOSING END

Figure 10: Hamlet as we know and love it

Cryptography

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 100 ER ENOHDLAE OHDLO UOZEOUNORU O UOZEO HD OITO HEOQSET IUROFHE HENO ITORUZAEN
 200 ES ELOHRNDE OHRNO UOVEOULOSU O UOVEO HR OITO HEOQAET IUSOPHE HELO ITOSUVDEL
 300 ES ELOHANDE OHANO UOVEOULOSU O UOVEO HA OITO HEOQRET IUSOFHE HELO ITOSUVDEL
 400 ES ELOHINME OHINO UOVEOULOSU O UOVEO HI OATO HEOQRET AUSOWHE HELO ATOSUVMEL
 500 ES ELOHINME OHINO UODEOULOSU O UODEO HI OATO HEOQRET AUSOWHE HELO ATOSUDMEL
 600 ES ELOHINME OHINO UODEOULOSU O UODEO HI OATO HEOQRET AUSOWHE HELO ATOSUDMEL
 900 ES ELOHANME OHANO UODEOULOSU O UODEO HA OITO HEOQRET IUSOWHE HELO ITOSUDMEL
1000 IS ILOHANMI OHANO RODIORLOSR O RODIO HA OETO HIOQUIT ERSOWHI HILO ETOSRDMIL
1100 ISTILOHANMITOHANOT ODIO LOS TOT ODIOTHATOEROTHIOQUIRTE SOWHITHILOTEROS DMIL
1200 ISTILOHANMITOHANOT ODIO LOS TOT ODIOTHATOEROTHIOQUIRTE SOWHITHILOTEROS DMIL
1300 ISTILOHARMITOHAROT ODIO LOS TOT ODIOTHATOENOTHIOQUINTE SOWHITHILOTENOS DMIL
1400 ISTILOHAMRITOHAMOT OFIO LOS TOT OFIOTHATOENOTHIOQUINTE SOWHITHILOTENOS FRIL
1600 ESTEL HAMRET HAM TO CE OL SOT TO CE THAT IN THE QUENTIOS WHETHEL TIN SOCREL
1700 ESTEL HAMRET HAM TO BE OL SOT TO BE THAT IN THE QUENTIOS WHETHEL TIN SOBREL
1800 ESTER HAMLET HAM TO BE OR SOT TO BE THAT IN THE QUENTIOS WHETHER TIN SOBLER
1900 ENTER HAMLET HAM TO BE OR NOT TO BE THAT IS THE QUESTION WHETHER TIS NOBLER
2000 ENTER HAMLET HAM TO BE OR NOT TO BE THAT IS THE QUESTION WHETHER TIS NOBLER
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Figure 11: Recovering Hamlet using MCMC!

References

Photos:

- Portraits taken from Wikipedia
- Letter distribution link: here
- Markov chain example link: here
- Gerrymandering link: here
- Cryptography link: here