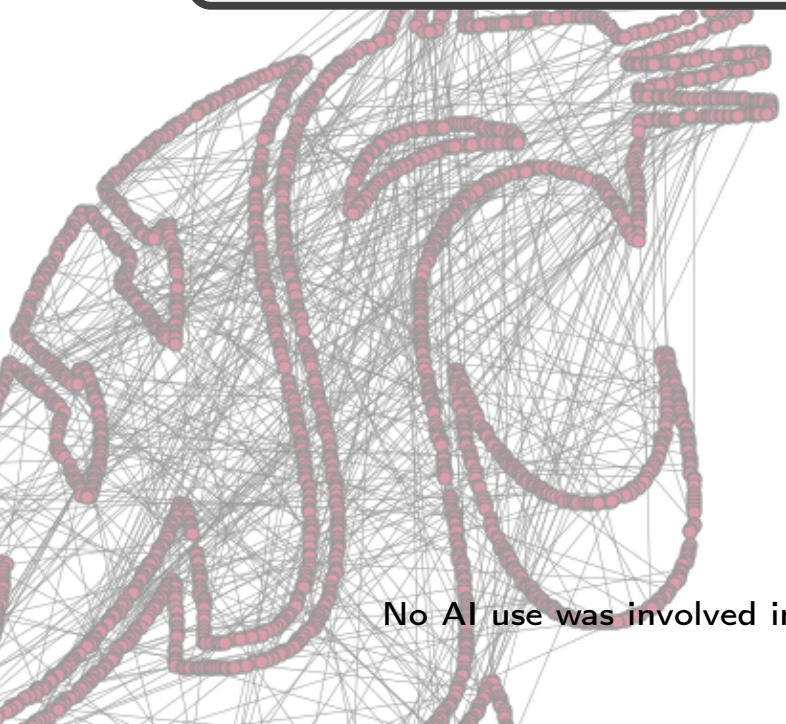
WASHINGTON STATE UNIVERSITY

When Graphs Share Eigenvectors



Garrett Kepler

October 12th, 2026

No AI use was involved in the creation of this work and the presenting slides

The Cats



Figure: On left, Beetle (stats: grey, loud, and 5 extra toes). On right, Nezuko (stats: black/orange, louder, and no extra toes).

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- Graphs
- Graph matrices

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- How do we think of graph eigenvectors?

General Results
- General perturbations
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Graph Introduction

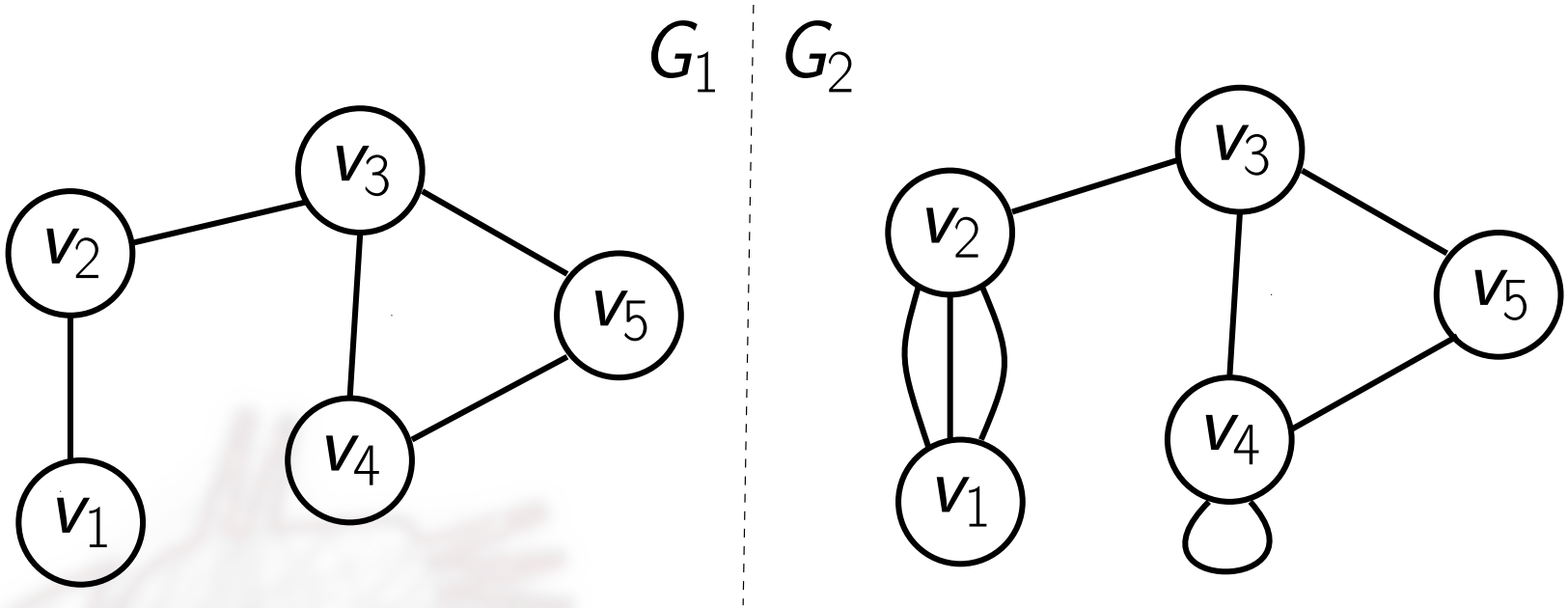
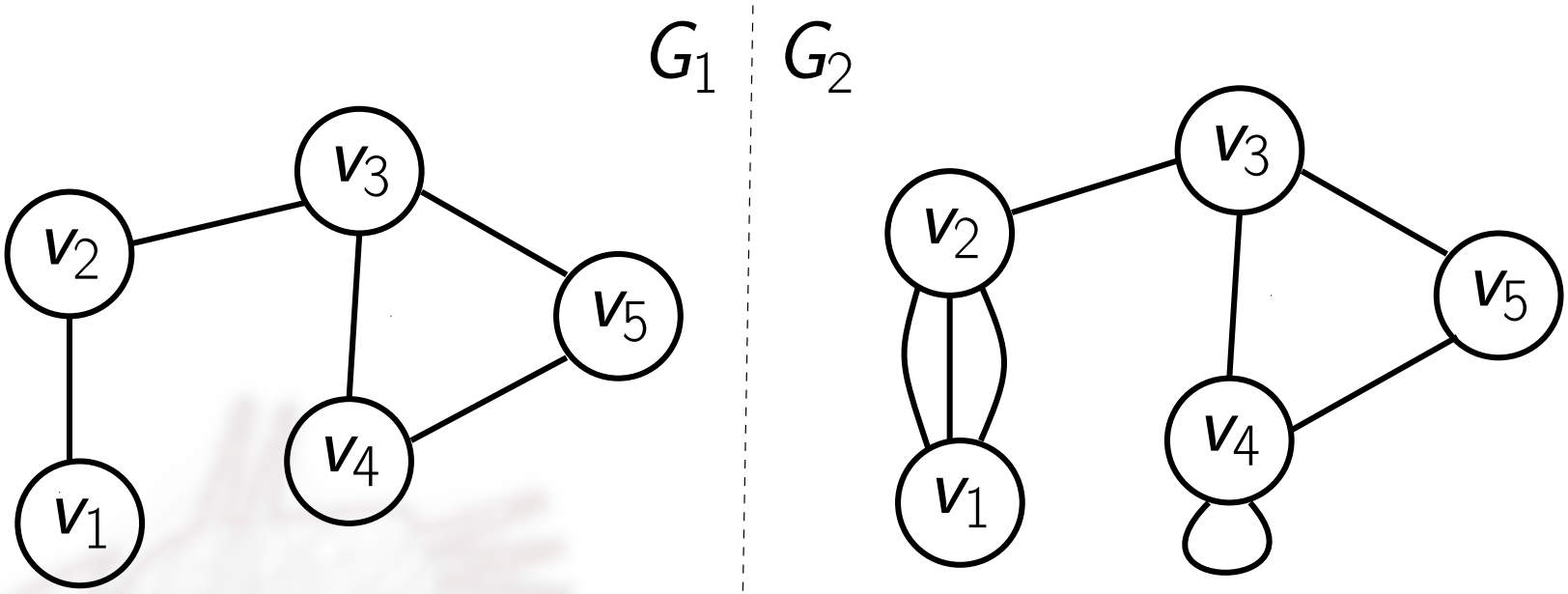


Figure: A couple graphs

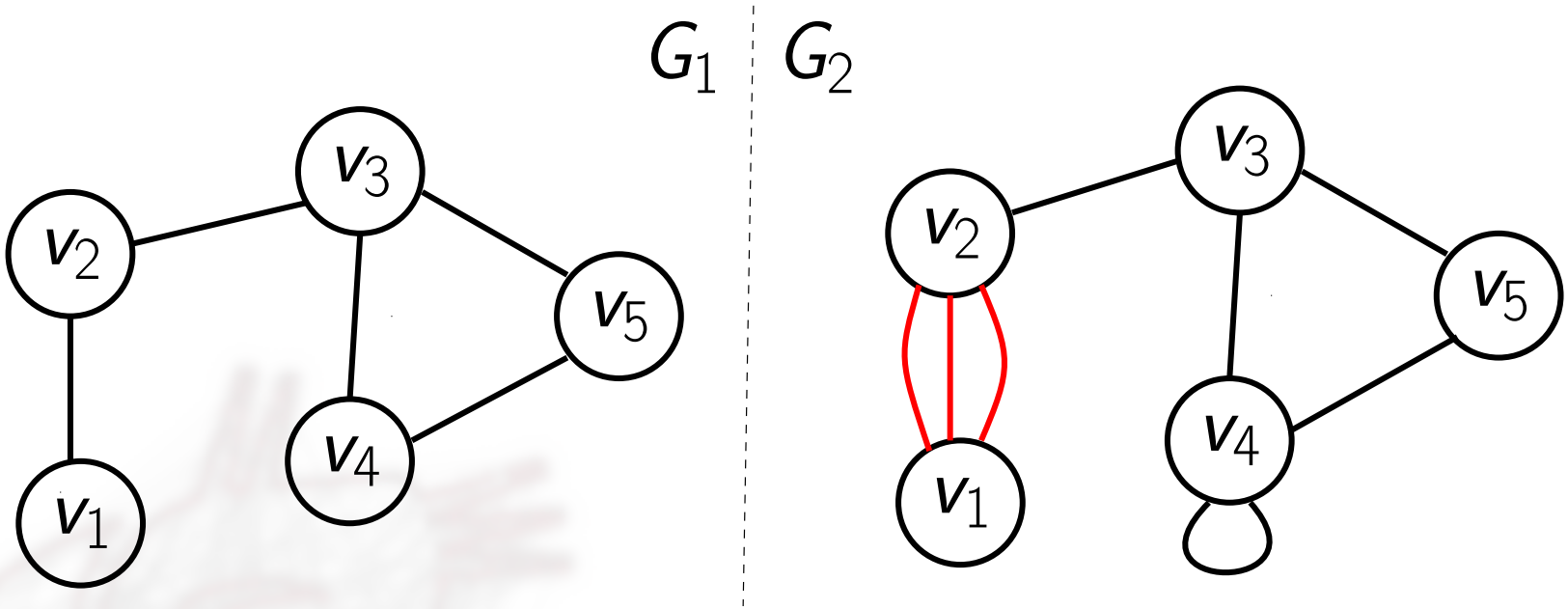
Definition

A graph $G = (V, E)$ consists of a set of vertices, V, with edges (unordered pairs) between them, E.



Note that G_1 and G_2 have the same set of vertices $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, but a different set of edges:

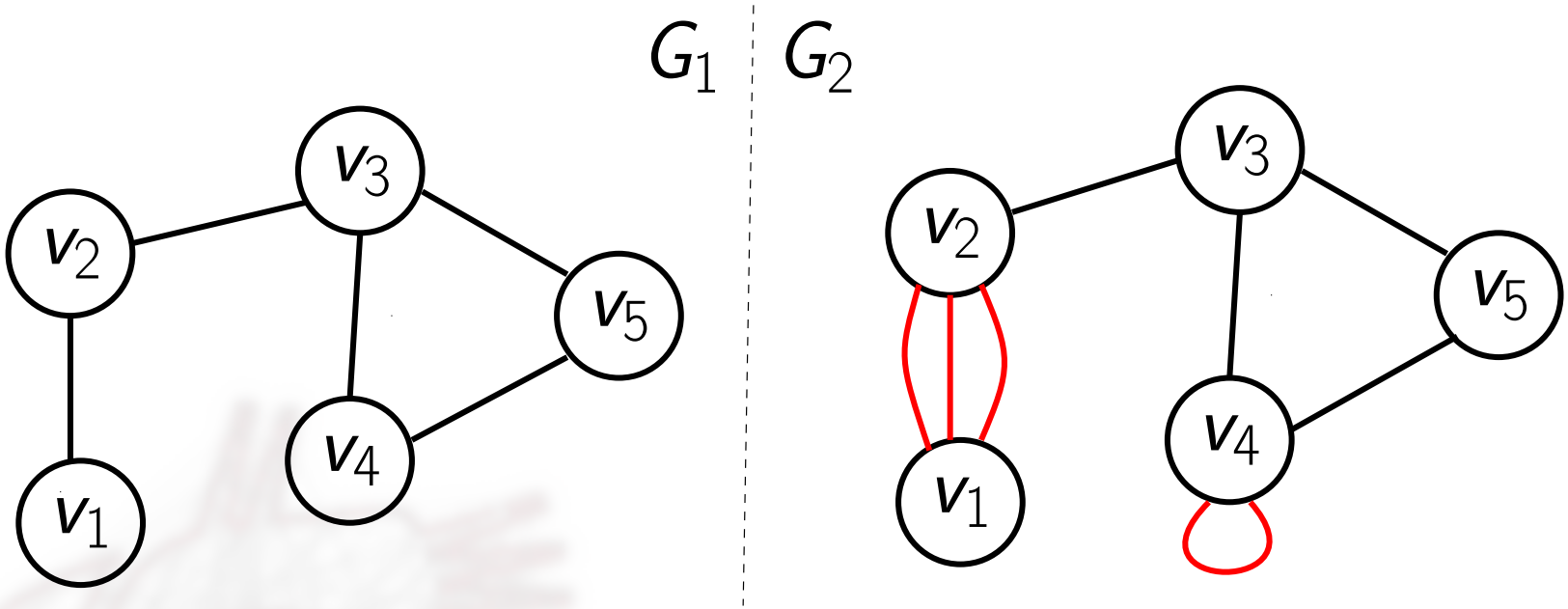
Graph Introduction



Note that G_1 and G_2 have the same set of vertices $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, but a different set of edges:

1. $\{v_1, v_2\}$ is repeated in G_2

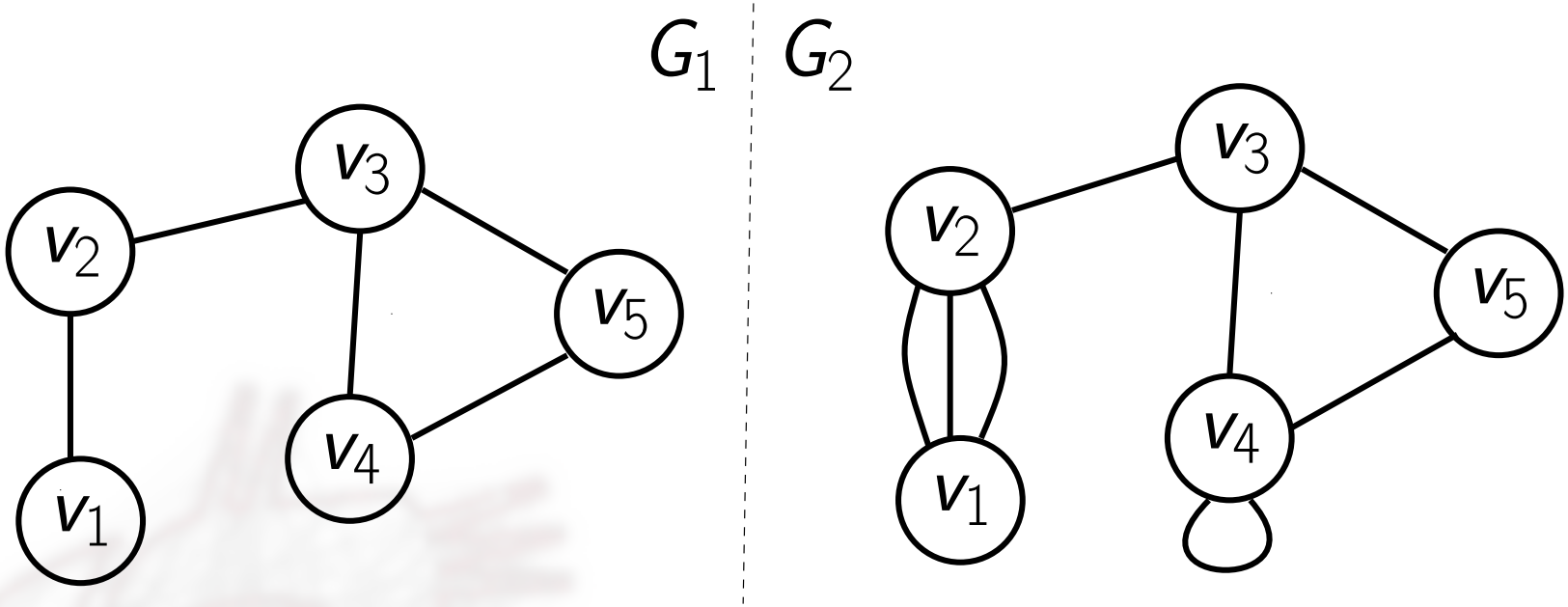
Graph Introduction



Note that G_1 and G_2 have the same set of vertices $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, but a different set of edges:

1. $\{v_1, v_2\}$ is repeated in G_2 but not in G_1
2. $\{v_4, v_4\}$ exists in G_2 but not G_1

Simple



We call G_1 *simple* and G_2 not simple:

Definition

A simple graph has at most one edge between vertices and no self-loops.

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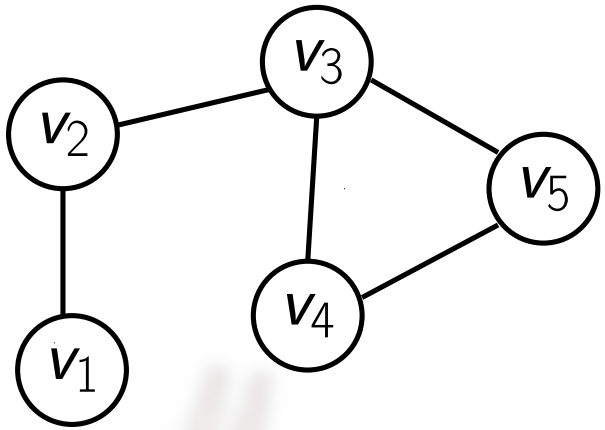
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Adjacency

G_1



$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

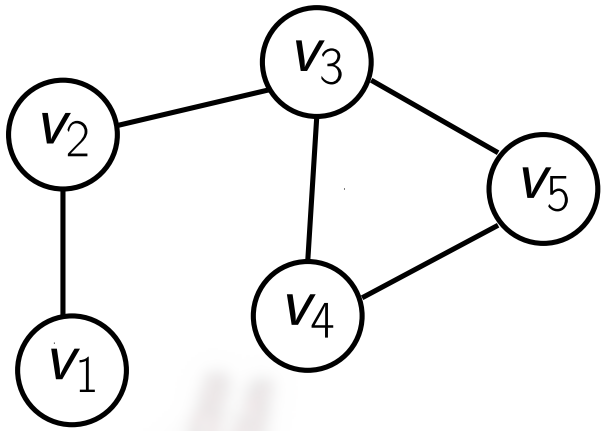
Definition

Let G be some simple graph. The *adjacency matrix* of a graph G, $A = A(G)$, has entries

$$A_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \text{ exists} \\ 0 & \text{if not} \end{cases}$$

Degree

G_1



$$D(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \end{array}$$

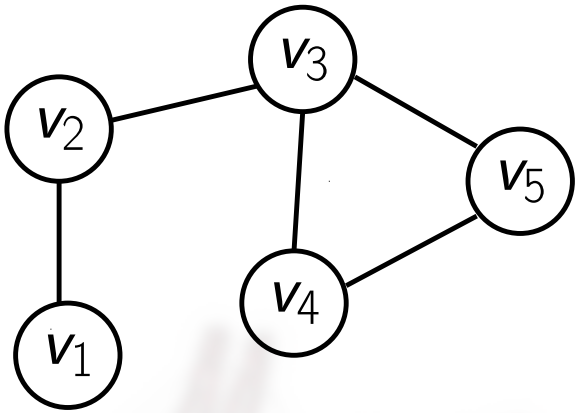
Definition

Let G be some simple graph and the degree of a vertex v, $d_G(v)$, be the number of edges with v in it. The *degree matrix* of a graph G, $D = D(G)$, has entries

$$D_{ij} = \begin{cases} d_G(i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Laplacian

G_1



$$L(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix} \end{array}$$

Definition

Let G be some simple graph. The *Laplacian matrix* of a graph G, $L = L(G)$, is simply $D - A$.

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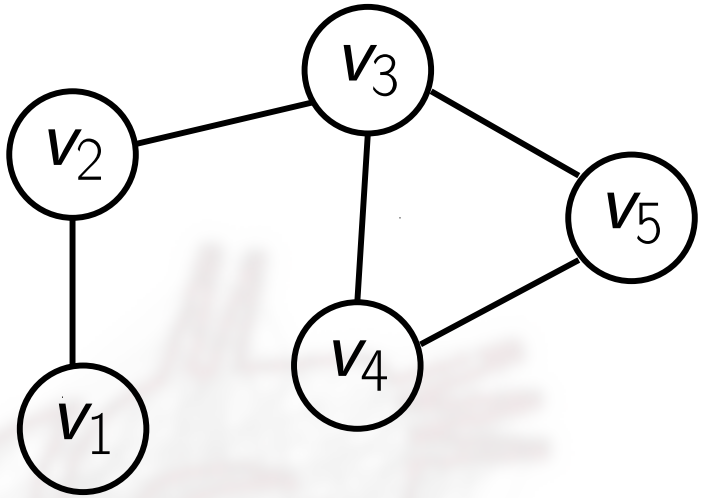
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Okay, so we've associated a matrix to the graph:

G_1



$$A(G_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

These matrices have eigenvectors. What do they look like for a graph?

Vectors

Consider the action of $A(G_1)$ on a vector $\vec{x} \in \mathbb{R}^5$

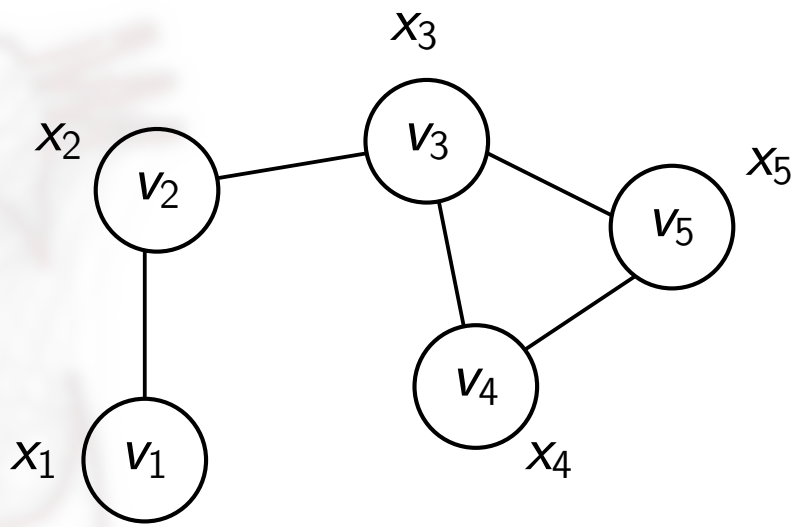
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + x_3 \\ x_2 + x_4 + x_5 \\ x_3 + x_5 \\ x_3 + x_4 \end{pmatrix}$$

This matrix encodes information about G_1. So, the matrix multiplication must be related to the structure of the graph. Let's explore how!

Vectors

Consider G_1. Using $\vec{x}$, we have assigned vertices a value.

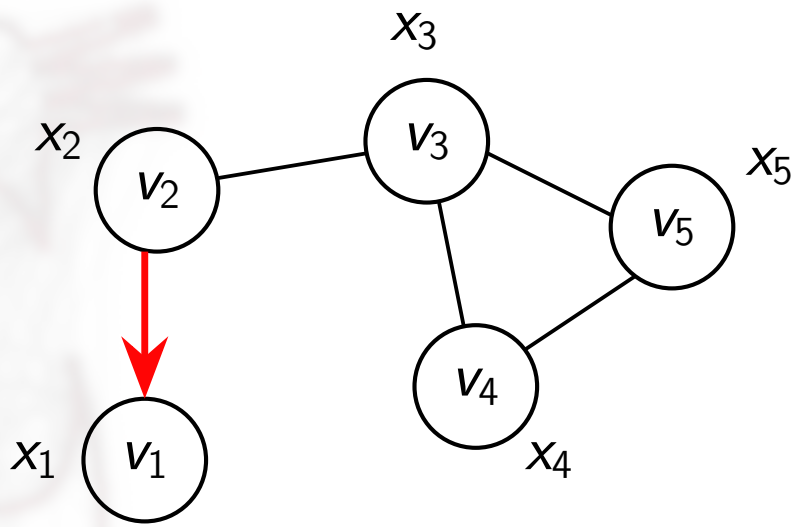
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + x_3 \\ x_2 + x_4 + x_5 \\ x_3 + x_5 \\ x_3 + x_4 \end{pmatrix}$$



Vectors

Our first step is to add up the values of the first vertex's neighbors.

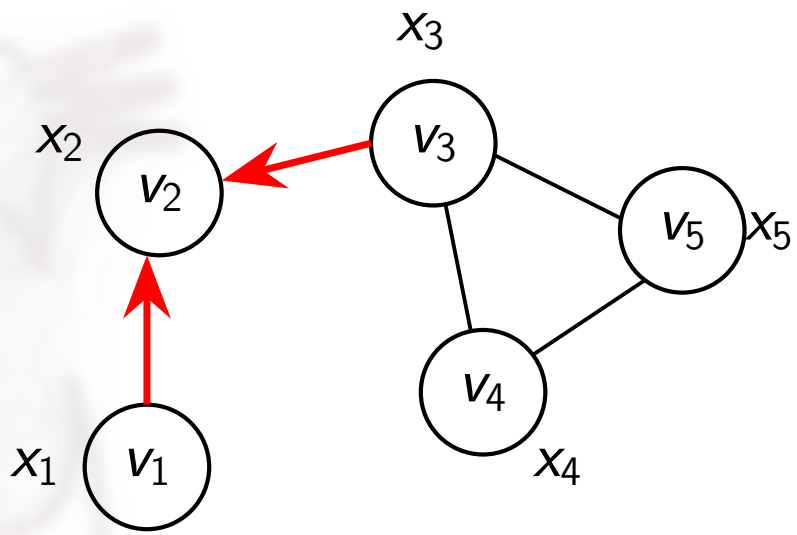
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_2 \\ \\ \\ \\ \end{pmatrix}$$



Vectors

Next, we add up the values of the second vertex's neighbors.

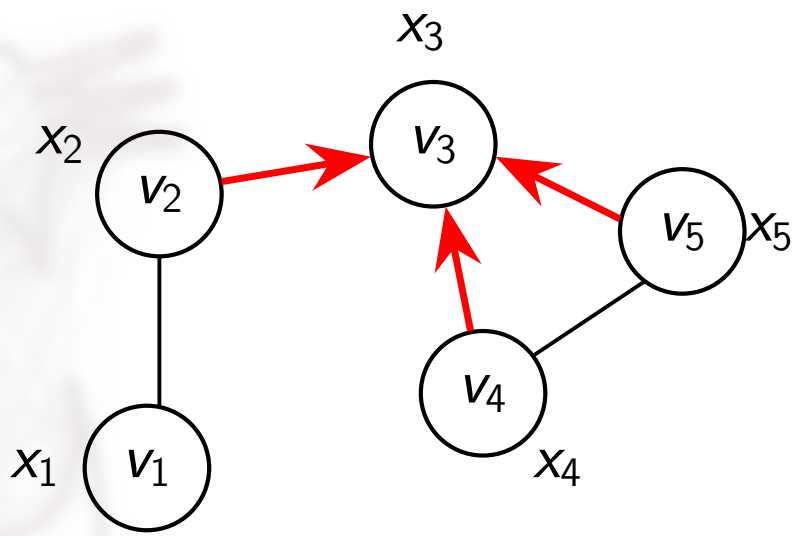
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + x_3 \\ \\ \\ \end{pmatrix}$$



Vectors

Next, the third vertex.

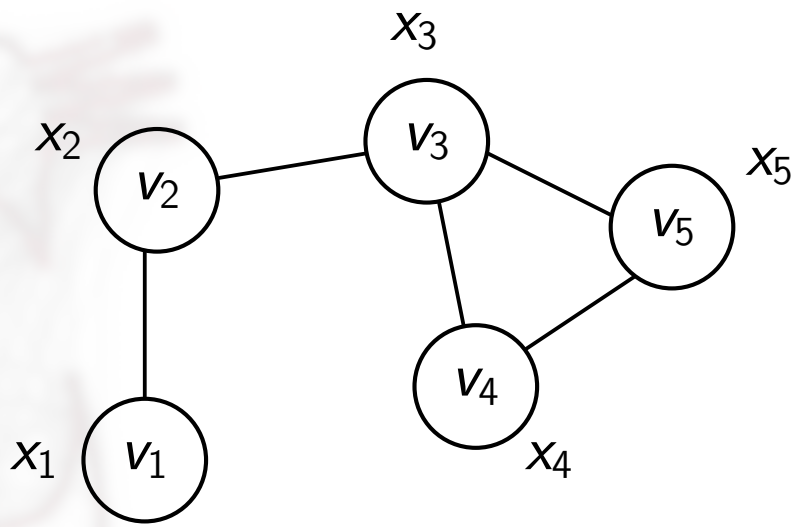
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + x_3 \\ x_2 + x_4 + x_5 \\ \\ \end{pmatrix}$$



Vectors

And we repeat for each vertex!

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 + x_3 \\ x_2 + x_4 + x_5 \\ x_3 + x_5 \\ x_3 + x_4 \end{pmatrix}$$



Vectors

We can think of a vector *assigning* vertices a value. The adjacency matrix then simply adds up the neighboring values for each vertex.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \sum_{\{1,j\} \in E} x_j \\ \sum_{\{2,j\} \in E} x_j \\ \sum_{\{3,j\} \in E} x_j \\ \sum_{\{4,j\} \in E} x_j \\ \sum_{\{5,j\} \in E} x_j \end{pmatrix}$$

Eigenvectors

If it is the case that these sums add up in a special way, we call $\vec{x}$ an eigenvector!

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \sum_{\{1,j\} \in E} x_j \\ \sum_{\{2,j\} \in E} x_j \\ \sum_{\{3,j\} \in E} x_j \\ \sum_{\{4,j\} \in E} x_j \\ \sum_{\{5,j\} \in E} x_j \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

A Local Argument

We want each equation to be satisfied for some λ. The Laplacian takes on a close, but different form:

Lemma

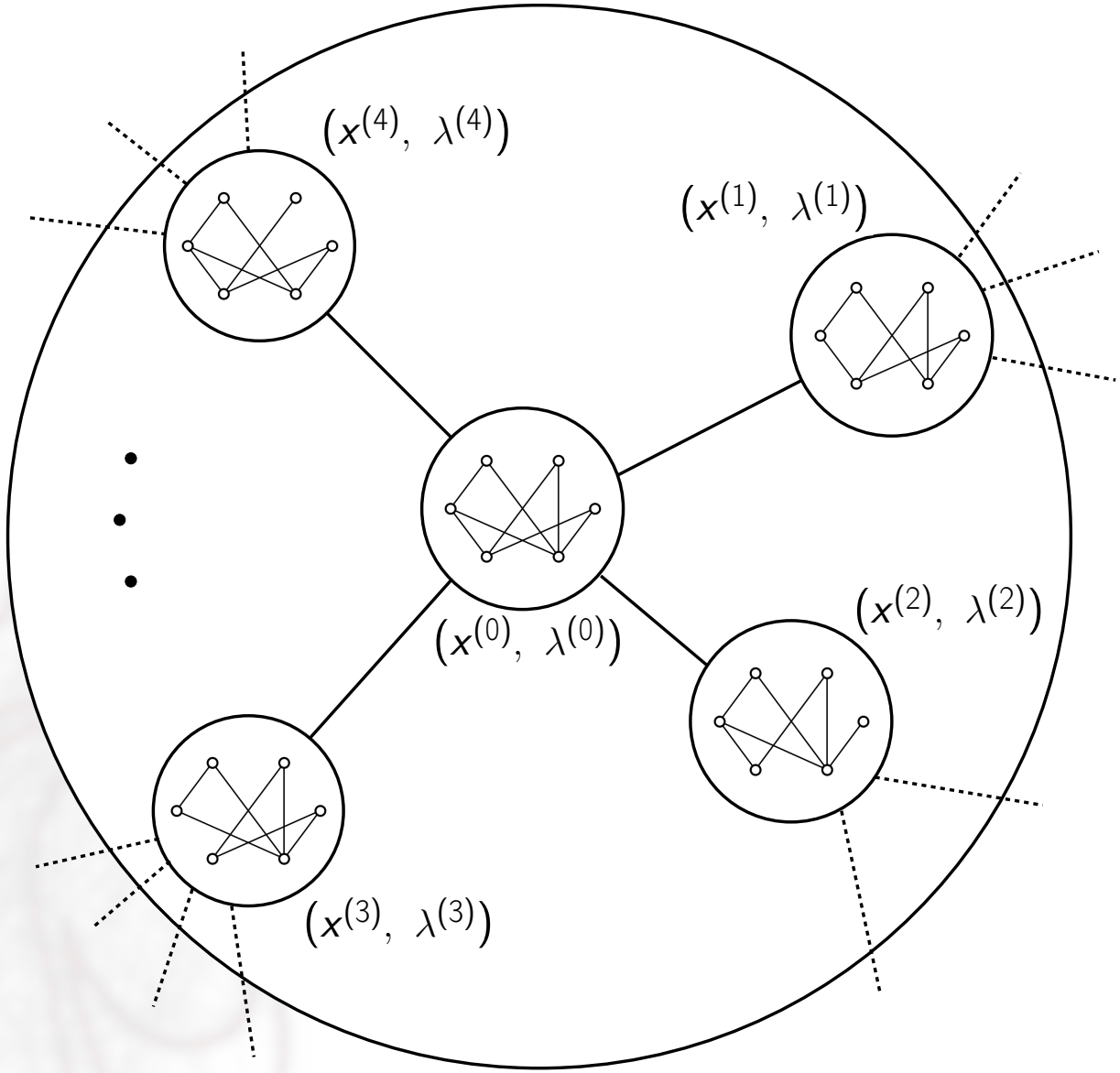
Let $G = (V, E)$ be a simple graph.

1. $A(G)\vec{x} = \lambda\vec{x} \iff \sum_{\{i,j\}\in E} x_j = \lambda x_i$ *for all* $i \in V$
2. $L(G)\vec{x} = \lambda\vec{x} \iff \sum_{\{i,j\}\in E} x_i - x_j = \lambda x_i$ *for all* $i \in V$

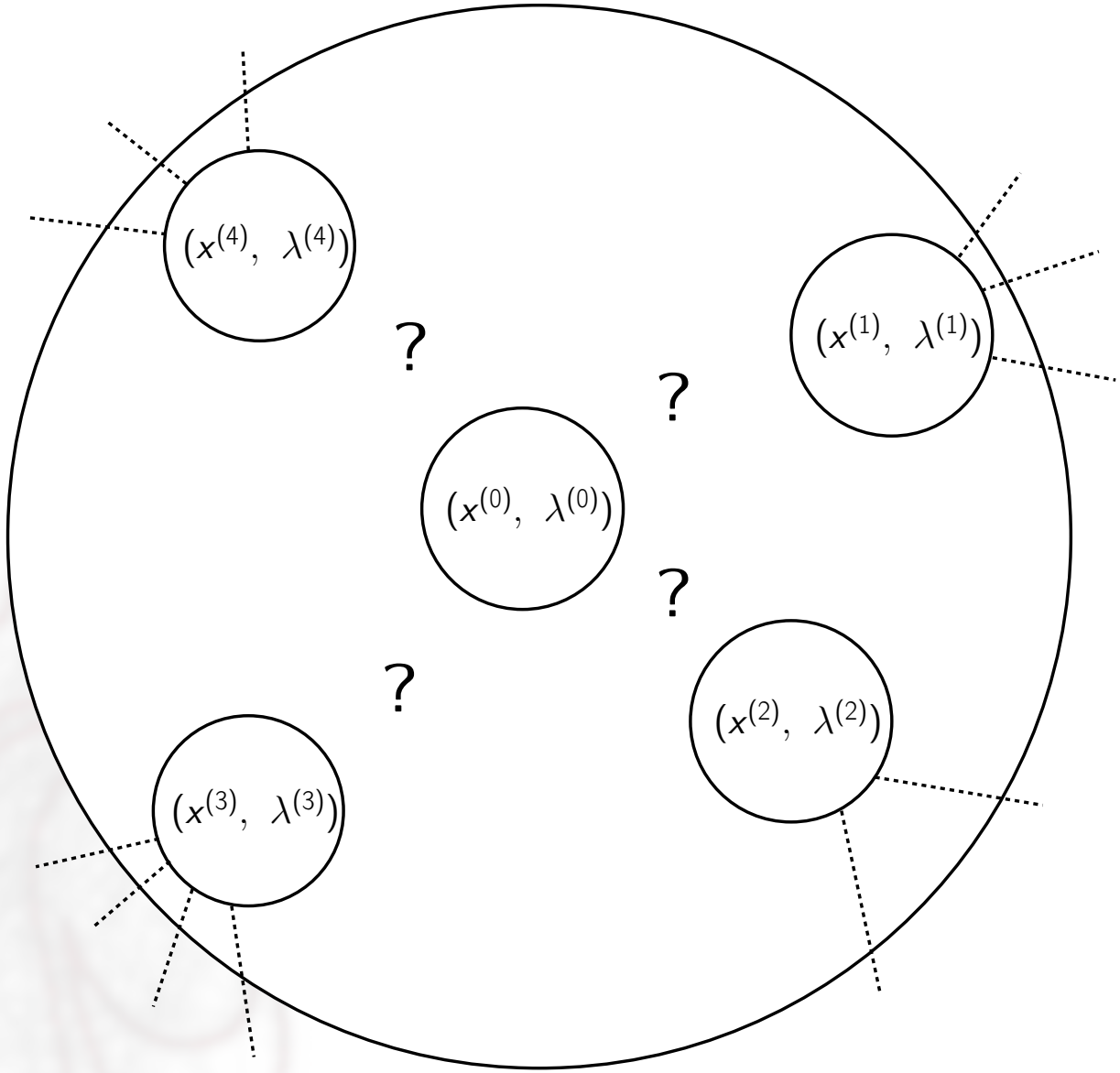
Question

Pretend we knew an eigenvector of a graph and we tweak the graph in some way. Do we know any spectral information of the new graph?

Metagraph



Metagraph



Interlude



Figure: Another cat nap

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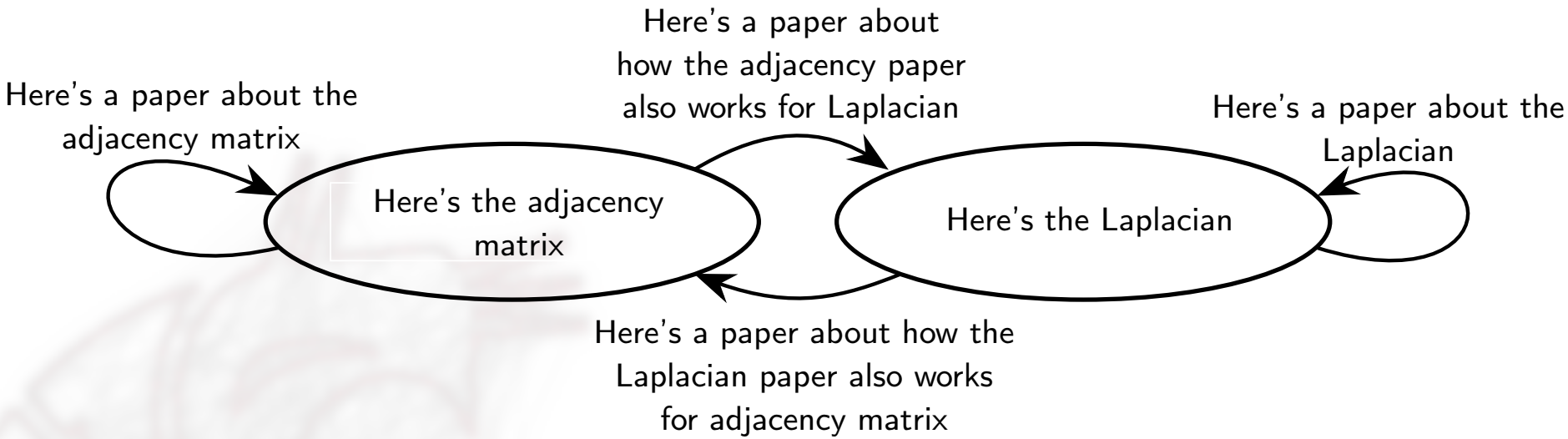
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A Fun Application to the Laplacian

Future Work

On Paper Writing

One quirk of the field:



Generalizing for Simplicity

I don't want to write (nor read) four papers! So, we prove stuff about a general matrix instead:

Definition

Let G be a simple graph. Then, we define $B_\alpha(G) = \alpha A(G) + (1-\alpha)L(G)$ where $\alpha \in [0,1]$. When α is arbitrary, we'll simply write B.

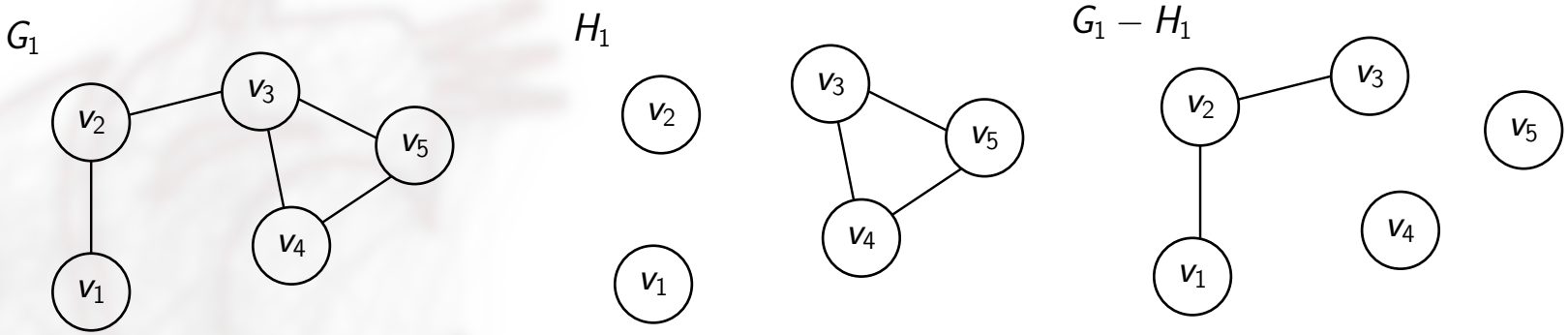
Observe:

1. $B_1 = A$
2. $B_0 = L$
3. $2B_{1/2} = D$
4. $[Bx]_i = \alpha \sum_{\{i,j\}\in E} x_j + (1-\alpha) \sum_{\{i,j\}\in E} x_i - x_j$
5. and more!

Remark

Consider removing the edges of some subgraph H of G to obtain $G - H$. We know two useful things:

1. $B(G) = B(G - H) + B(H)$
2. Suppose $B(G)x = \lambda x$. Then, $B(G - H)x = \lambda' x$ if and only if $B(H)x = (\lambda - \lambda')x$



Basics

Theorem

Let $B(G)x = \lambda x$ and λ' be some scalar. Then, the following are equivalent:

1. $B(H)x = \lambda' x$
2. $B(G - H)x = (\lambda - \lambda')x$
3. $\alpha \sum_{\{i,j\} \in E(H)} x_j + (1 - \alpha) \sum_{\{i,j\} \in E(H)} x_i - x_j = \lambda' x_i$ *for all* i
4. $\alpha \sum_{\{i,j\} \in E(G) \setminus E(H)} x_j + (1 - \alpha) \sum_{\{i,j\} \in E(G) \setminus E(H)} x_i - x_j = (\lambda - \lambda')x_i$ *for all* i

Corollary

Let $B(G)x = \lambda x$. Then, $B(H)x = \lambda x$ if and only if $B(G - H)x = 0$.

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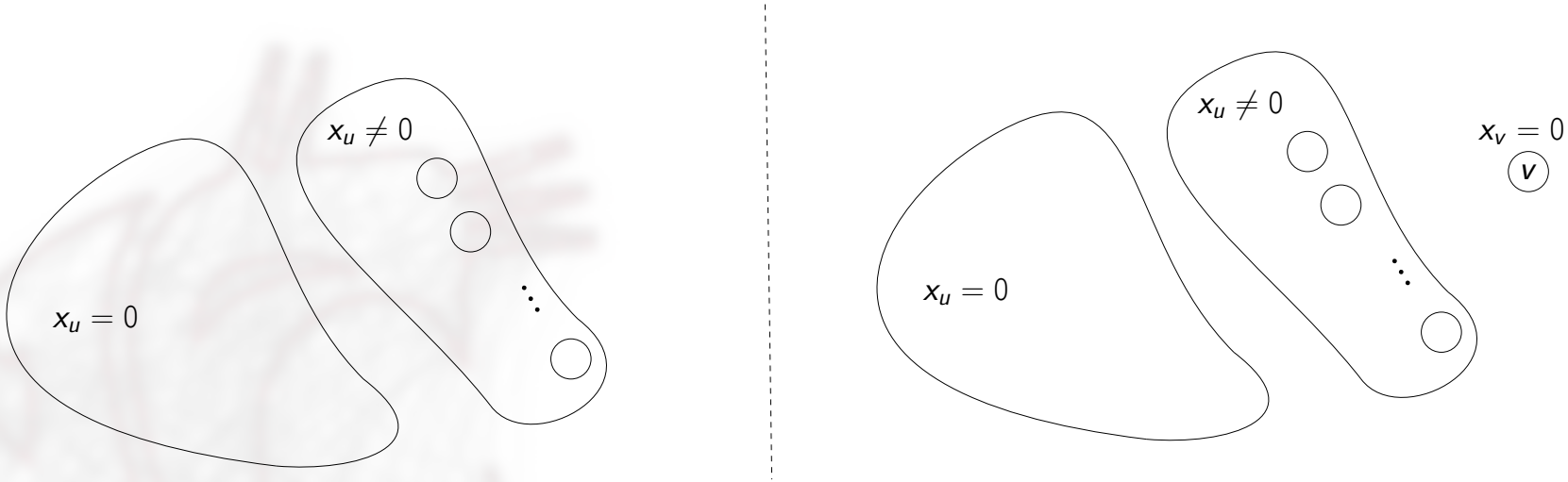
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Future Work

We can add a vertex!

Theorem (1)

Let $B(G)x = \lambda x$. Consider adding a vertex v to G with no additional edges to obtain $G \cup \{u\}$. Then, x can be extended to an eigenvector $y = [x, 0]^T$ of $B(G \cup \{u\})$ corresponding to λ.

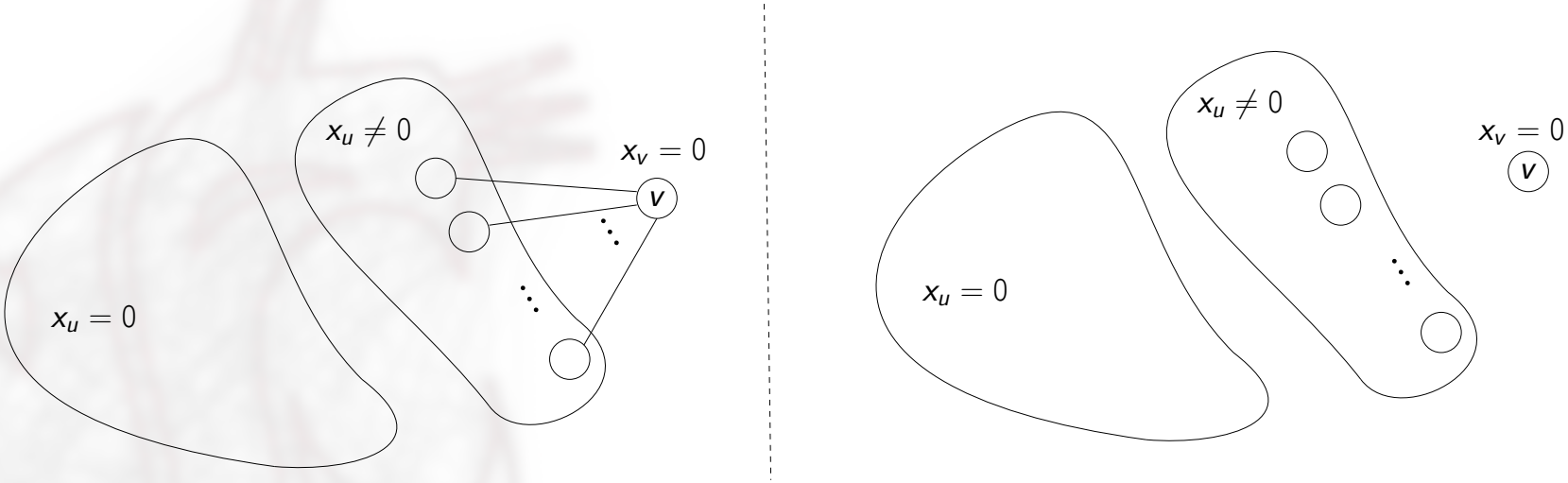


(x, λ) *eigenpair* $\implies$ (x, λ) *eigenpair*

We can remove edges to that vertex!

Theorem (2)

Let N_v be the set of edges in a graph that contain a vertex v. Consider a vector $x \in \mathbb{R}^n$ such that $x^T \mathbb{1} = 0$ and $x_v = 0$ for some $v \in V(G)$. If v is adjacent to every vertex $u \in V(G)$ such that $x_u \neq 0$, then $B(G)x = \lambda x$ if and only if $B(G - N_v)x = (\lambda - 1 + \alpha)x$.

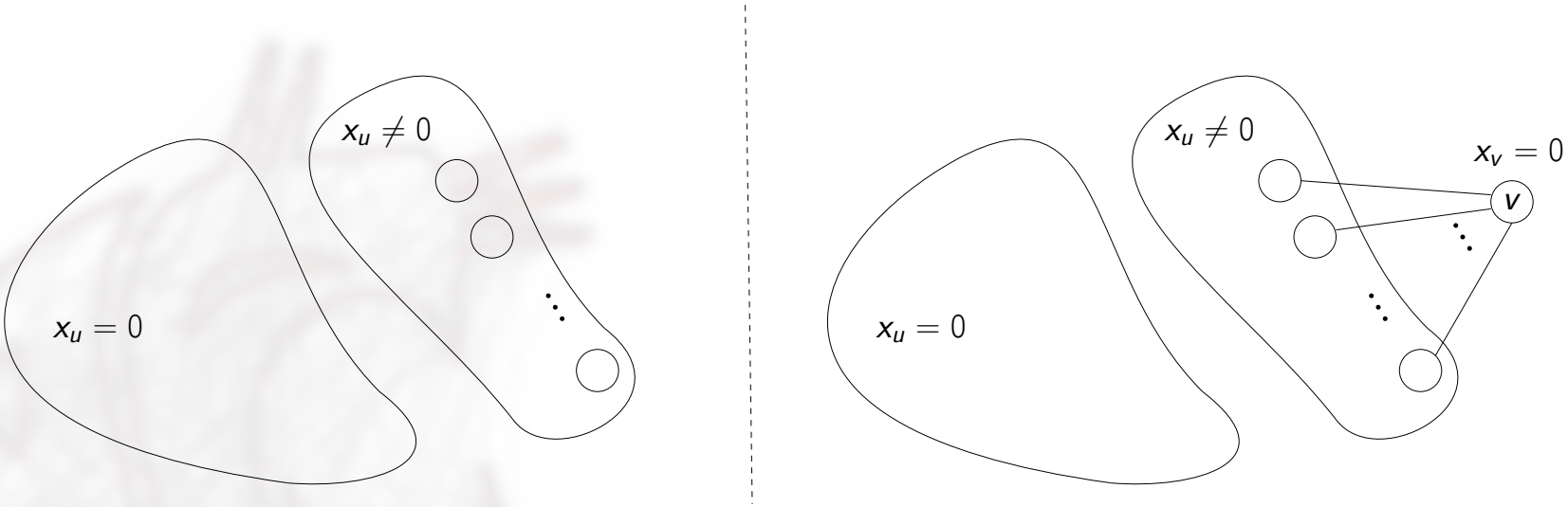


(x, λ) *eigenpair* $\iff$ $(x, \lambda - 1 + \alpha)$ *eigenpair*

We can remove that vertex entirely!

Corollary (3)

Let $Bx = \lambda x$ such that $x^T \mathbb{1} = 0$. Consider adding a vertex u with edges between u and every vertex in $v \in V(G)$ such that $x_v \neq 0$. If G' is this new graph, then x can be extended to an eigenvector $y = [x, 0]^T$ of $B(G')$ corresponding to $\lambda - 1 + \alpha$.



(x, λ) *eigenpair* $\implies$ $(x, \lambda - 1 + \alpha)$ *eigenpair*

On Single Edge Removal

$$B(G)x = \lambda x$$
$$\iff B(G - \{u, v\})x + B(\{u, v\})x = \lambda x$$

What happens $B(G - \{u, v\})$ also has eigenvector x?

$$B(G - \{u, v\})x = \lambda' x \implies \lambda' x + B(\{u, v\})x = \lambda x$$
$$\implies B(\{u, v\})x = (\lambda - \lambda')x$$

Luckily, $B(\{u, v\})x$ is straight forward to compute:

$$[B(\{u, v\})x]_i = \begin{cases} 0 & i \neq u, v \\ \alpha x_u + (1 - \alpha)(x_v - x_u) & i = v \\ \alpha x_v + (1 - \alpha)(x_u - x_v) & i = u \end{cases}$$

On Single Edge Removal

If we know $B(\{u,v\})x = (\lambda - \lambda')x$, then we are asking when the following holds:

$$[B(\{u,v\})x]_i = \begin{cases} 0 & i \neq u,v \\ \alpha x_u + (1-\alpha)(x_v - x_u) & i = v \\ \alpha x_v + (1-\alpha)(x_u - x_v) & i = u \end{cases}$$

$$= \begin{cases} (\lambda - \lambda')x_i & i \neq u,v \\ (\lambda - \lambda')x_v & i = v \\ (\lambda - \lambda')x_u & i = u \end{cases}$$

Cases to consider:

1. $\lambda = \lambda'$
2. $\lambda \neq \lambda'$

Results on Single Edge Removal

Theorem (4)

Let (x, λ) be an eigenpair of $B(G)$ for a non-empty graph G. Then, for any edge $\{u, v\} \in E(G)$,

1. *(x, λ) is an eigenpair of $B(G - \{u, v\})$ if and only if $(1 - \alpha)(x_u - x_v) = -\alpha x_v = \alpha x_u$.*
2. *(x, λ') is an eigenpair of $B(G - \{u, v\})$ where $\lambda' \neq \lambda$ if and only if $x_i = 0$ for all $i \neq u, v$ and $x_u = -x_v \neq 0$. Moreover, the following must hold:*
   - 2.1 *u and v must have the same degree*
   - 2.2 $\lambda' = (1 - \alpha)(d_G(u) - 1) = (1 - \alpha)(d_G(v) - 1)$
   - 2.3 $\lambda = \lambda' + 2 - 3\alpha = \lambda' + 2 - 3\alpha$

Interlude

We'll now explore the properties of eigenvector sharing between graphs differing by a single edge.

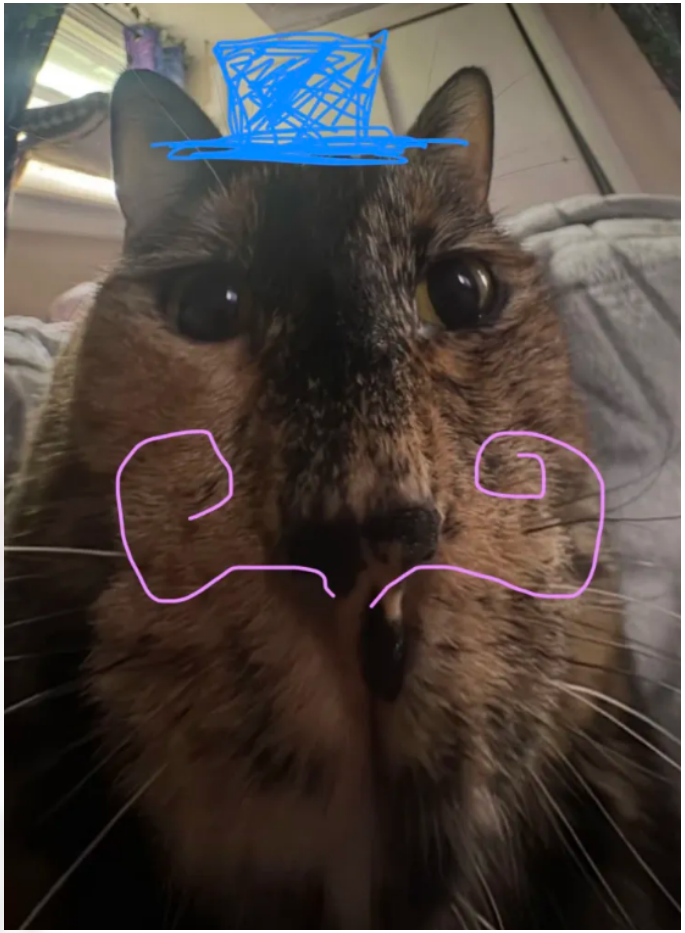


Figure: Nezuko with a top hat and mustache (top hat and mustache are not real)

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A Fun Application to the Laplacian

Future Work

Which graphs?

Question

Consider the second case "(x, λ') is an eigenpair of $B(G - \{u, v\})$ where $\lambda' \neq \lambda$ if and only if $x_i = 0$ for all $i \neq u, v$ and $x_u = -x_v \neq 0$". The restrictions that

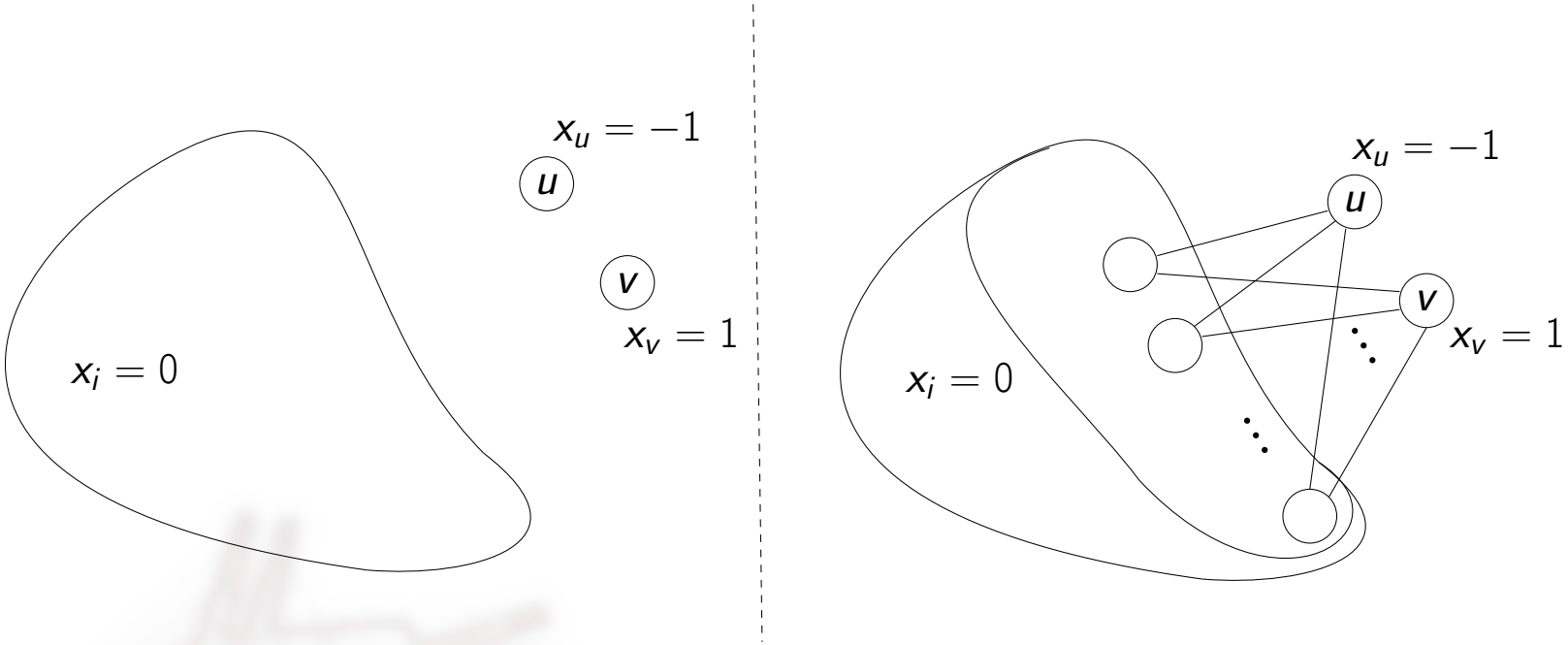
1. u and v must have the same degree
2. $\lambda' = (1 - \alpha)(d_G(u) - 1) = (1 - \alpha)(d_G(v) - 1)$
3. $\lambda = \lambda' + 2 - 3\alpha$

is extremely specific. What do these graphs look like?

Intuition:

- For $x_i = 0$ and $[Bx]_i = 0$, we need i to be connected to both u and v or neither (Theorem 2).

Which graphs?



x eigenvector $\iff$ *x eigenvector*

Theorem

The graphs admitting the eigenvectors $x \in \mathbb{R}^n$ *with all zeroes except for two distinct entries* $x_u = -x_v \neq 0$ *are formed by adding vertices and edges to a graph with two vertices u and v such that each other vertex is either adjacent to both u and v or neither u nor v.*

How many graphs?

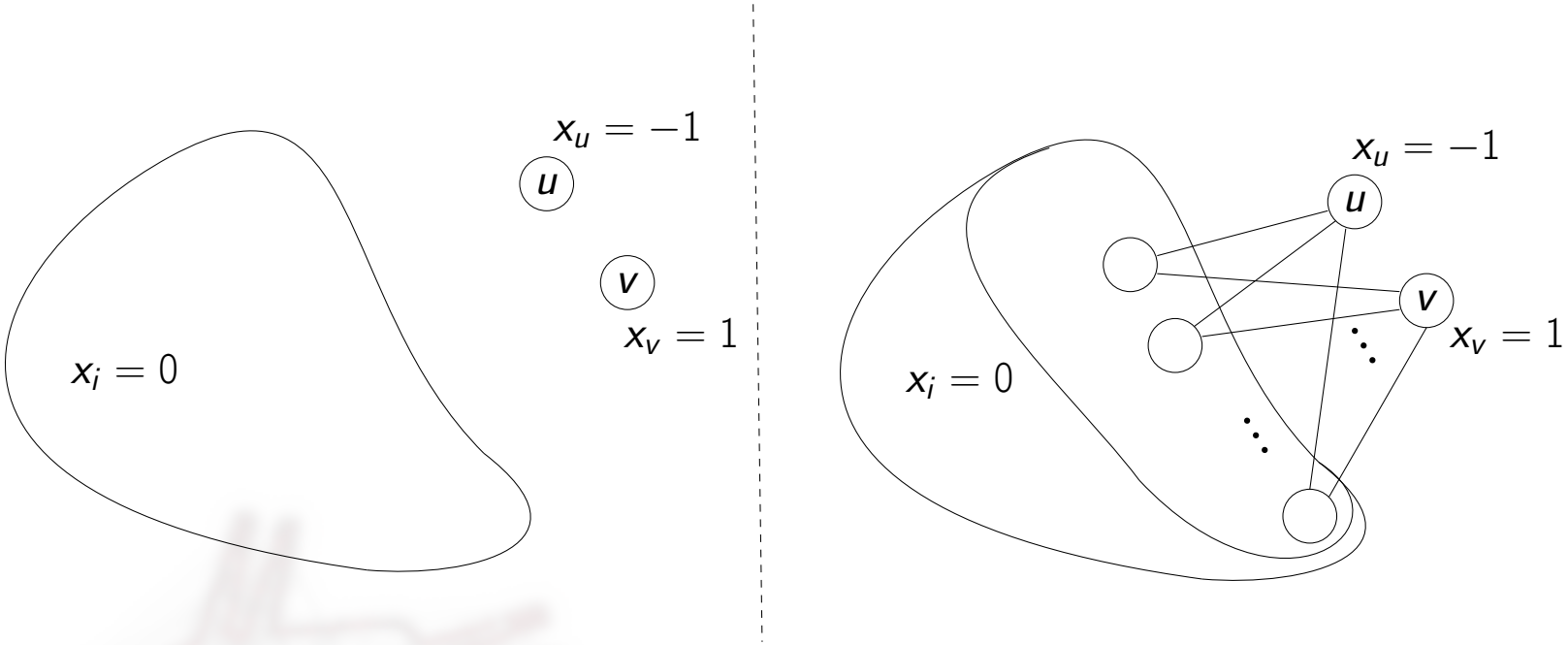
Question

Okay, but this seems incredibly restrictive. Does this type of graph ever appear in practice?

Intuition: Take a simple graph on n vertices and use the previous theorems to count!

1. Pick 2 vertices, u and v. Assign one vertex 1 and the other -1. There are $\binom{n}{2}$ ways to do this.
2. Assign the other $n-2$ vertices 0.
3. Connect these other vertices to either both u and v or neither. There are 2^{n-2} ways to form these connections.
4. There are $2^{\binom{n-2}{2}}$ ways to form a subgraph on the other vertices.
5. Either u and v are connected or not: 2 ways.

How many graphs?



There are $2^{\binom{n}{2}}$ possible simple graphs, and $\binom{n}{2} \cdot 2^{n-2} \cdot 2^{\binom{n-2}{2}} \cdot 2$ possible simple graphs with these eigenvectors. Taking the ratio (and using L'Hopital's rule finally), we can show:

Theorem

The proportion of graphs admitting eigenvectors $x \in \mathbb{R}^n$ with all zeroes except for two distinct entries $x_u = -x_v \neq 0$ approaches 1 as $n \to \infty$.

Background
- Graphs
- Graph matrices

Motivation
- How do we think of graph eigenvectors?

General Results
- General perturbations
- Structural Properties

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Sharing all eigenvectors

Setting $\alpha = 0$, Theorem 4 about B_α is converted to the Laplacian:

Corollary

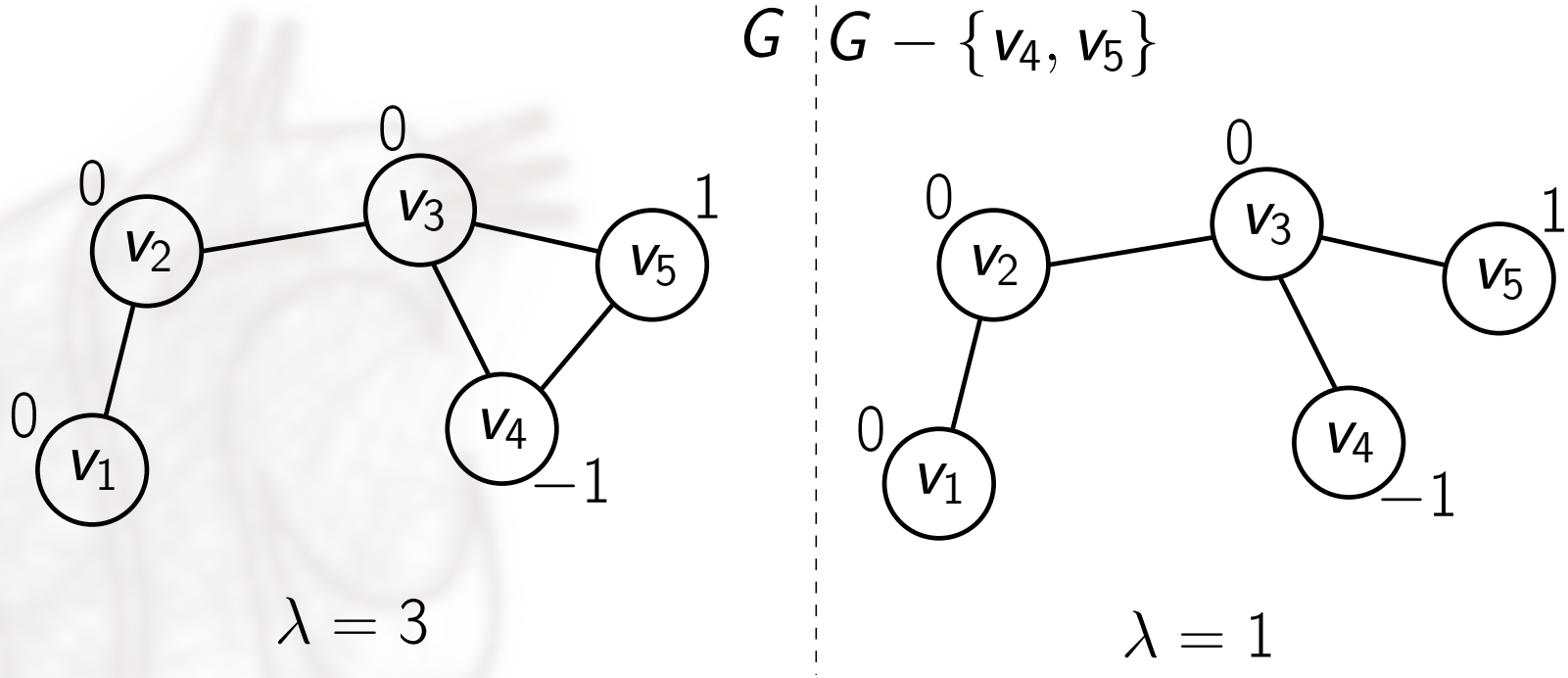
Let G be a non-empty, undirected graph such that $L(G)x = \lambda x$. Then, for any $\{u, v\} \in E(G)$,

1. *$L(G - \{u, v\})x = \lambda x$ if and only if $x_u = x_v$.*
2. *$L(G - \{u, v\})x = \lambda' x$ where $\lambda' \neq \lambda$ if and only if $x_i = 0$ for all $i \neq u, v$ and $x_u = -x_v \neq 0$. Moreover,*
   - 2.1 *u and v have the same degree*
   - 2.2 $\lambda' = d_G(u) - 1 = d_G(v) - 1$
   - 2.3 $\lambda = \lambda' + 2$

Sharing all eigenvectors

So, in order for $L(G)$ and $L(G - \{u, v\})$ to share an eigenvector, either $x_u = x_v$ or $x_u = -x_v \neq 0$ and everywhere else is zero.

Moreover, in the second case, we know which graphs will admit this kind of eigenvector. So, we can construct graphs that share a *single* eigenvector.



Sharing all eigenvectors

Question

Can we make graphs that share *all* eigenvectors?

Consider the second case: $\lambda = d_G(u) + 1 = d_G(v) + 1$
$L(G)x = \lambda x$ where $x_u = -x_v \neq 0$ and zeroes elsewhere.

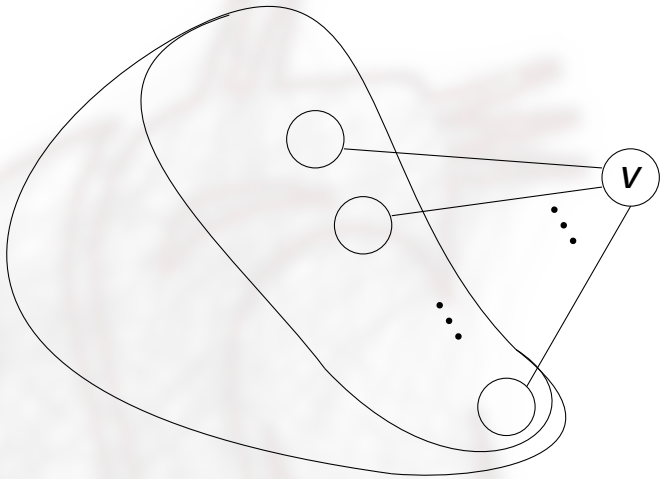
- $L(G)$ is symmetric, so $x^T y = 0$ for any y such that $Ly = \lambda' y$ $(\lambda \neq \lambda')$.
- $x^T y = 0 \implies x_u y_u + x_v y_v = 0 \implies y_u - y_v = 0 \implies y_u = y_v$ (the first case!)
- So, both x and y are shared by $L(G)$ and $L(G - \{u, v\})$.

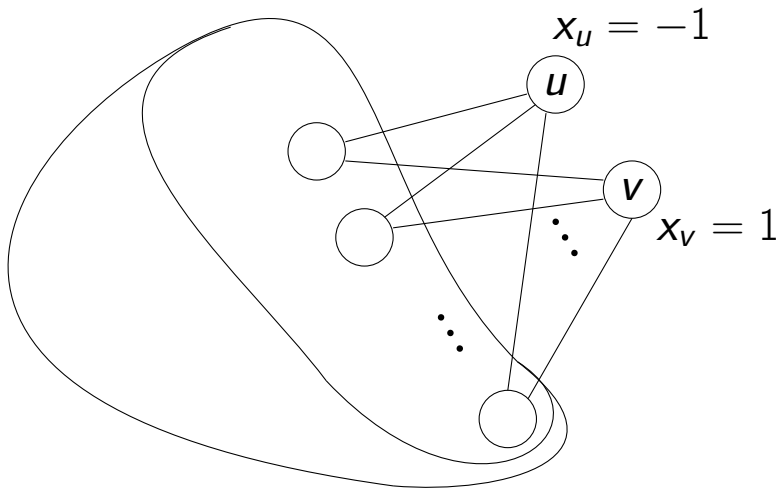
Sharing all eigenvectors

A similar argument applies to $L(G - \{u, v\})$ such that:

Theorem

If G admits Laplacian eigenvector $x_i = 0$ for all $i \neq u, v$ and $x_u = -x_v \neq 0$ corresponding to $\lambda = d_G(u) + 1 = d_G(v) + 1$, then $L(G)$ and $L(G - \{u, v\})$ share all eigenvectors.

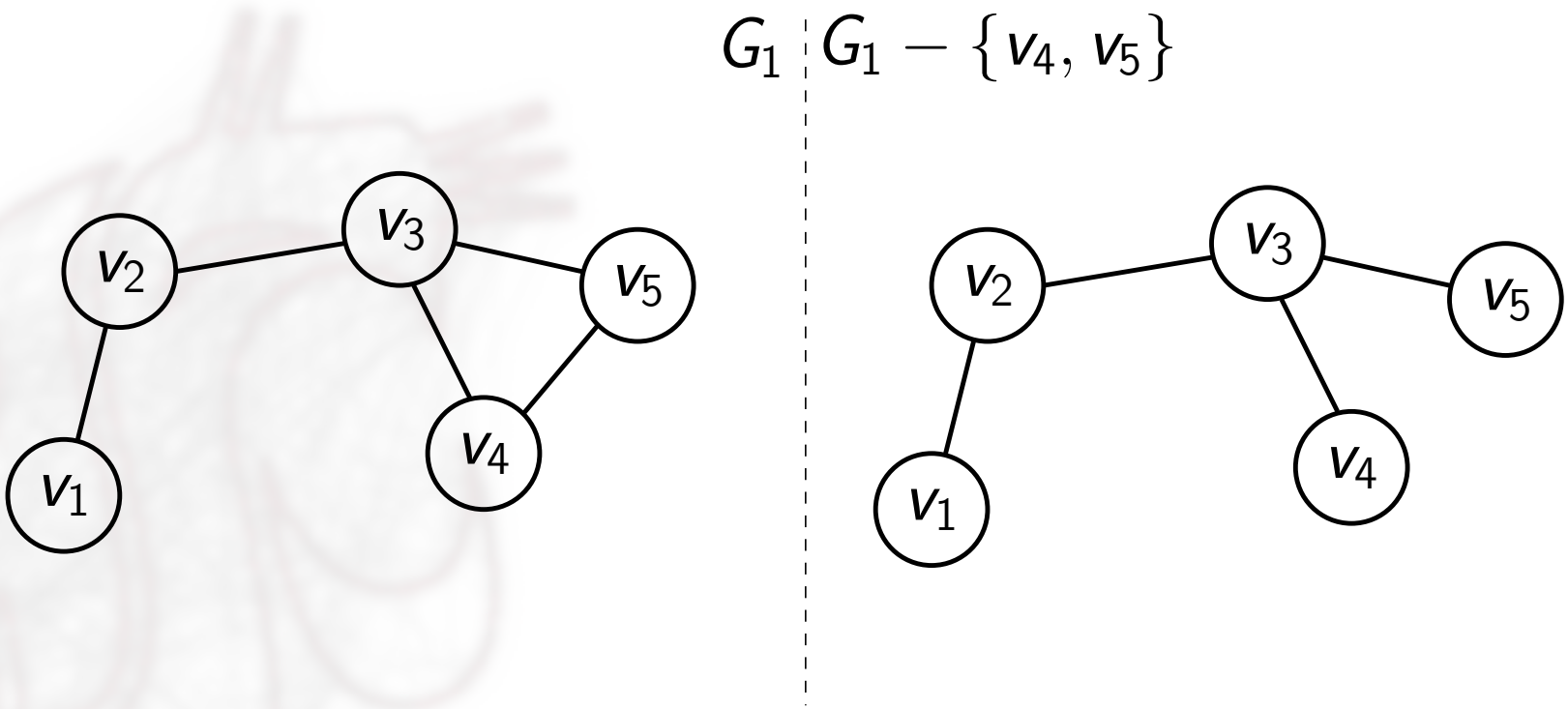




Sharing all eigenvectors

Numpy told me these are the eigenvectors of $L(G_1)$:

$$\begin{pmatrix} -0.447 & -0.702 & -0.536 & 0 & 0.138 \\ -0.447 & -0.338 & 0.703 & 0 & -0.438 \\ -0.447 & 0.202 & 0.318 & 0 & 0.811 \\ -0.447 & 0.419 & -0.242 & 0.707 & -0.256 \\ -0.447 & 0.419 & -0.242 & -0.707 & -0.256 \end{pmatrix}$$



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Sharing all adjacency eigenvectors?

Since $B_1 = A$, we obtain the following corollary:

Corollary

Let G be a graph with at least one edge such that $A(G)x = \lambda x$. For any $\{u, v\} \in E(G)$,

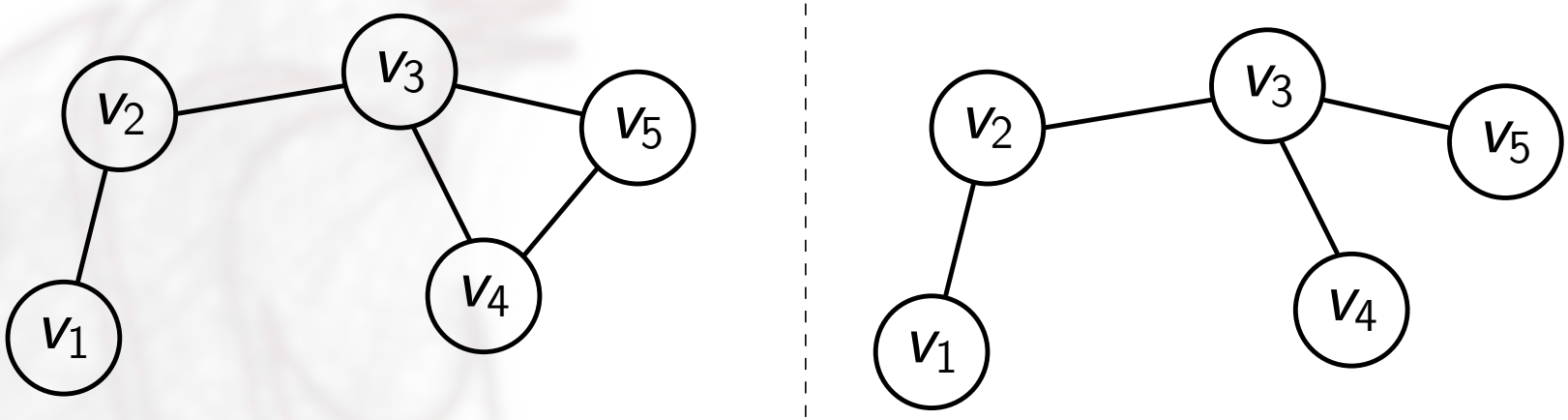
1. *$A(G - \{u, v\})x = \lambda x$ if and only if $x_u = x_v = 0$*
2. *$A(G - \{u, v\})x = \lambda' x$ where $\lambda' \neq \lambda$ if and only if $x_i = 0$ for all $i \neq u, v$ and $x_u = -x_v \neq 0$. Moreover,*
   - 2.1 *u and v have the same degree*
   - 2.2 $\lambda = -1$
   - 2.3 $\lambda' = 0$

Sharing all adjacency eigenvectors?

Numpy told me these are the eigenvectors of $A(G_1)$:

$$\begin{pmatrix} 0.354 & 0 & -0.671 & -0.632 & -0.155 \\ -0.593 & 0 & 0.362 & -0.632 & -0.342 \\ 0.639 & 0 & 0.476 & 0 & -0.604 \\ -0.239 & -0.707 & -0.309 & 0.316 & -0.497 \\ -0.239 & 0.707 & -0.309 & 0.316 & -0.497 \end{pmatrix}$$

G_1 | $G_1 - \{v_4, v_5\}$



Sharing all adjacency eigenvectors?

Question

How do we make graphs G_1 and G_2 such that $A(G_1)$ and $A(G_2)$ share *all* eigenvectors? [Kepler, Roberson, Wilson 2027+]

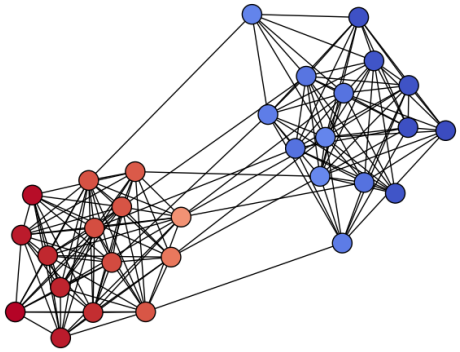
Answer (*)

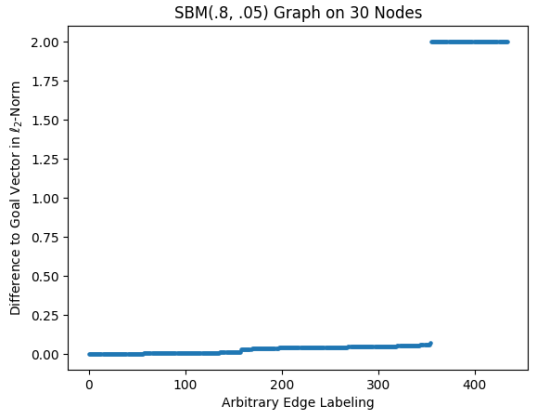
If G is connected, $A(G)$ and $A(G - \{u, v\})$ will not share *all* eigenvectors.

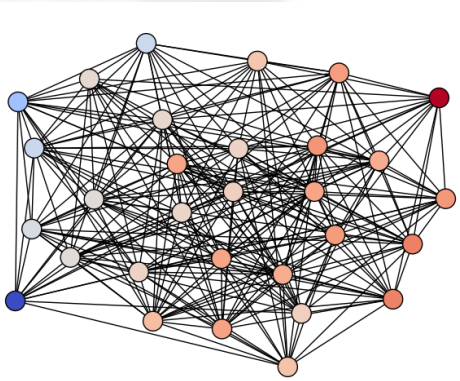
What about literally any other eigenvector case?

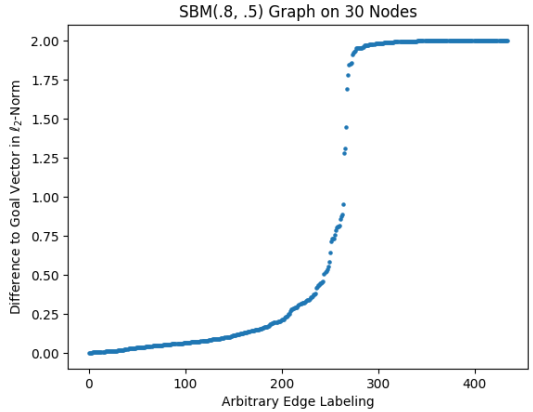
Question

Okay, we know when the share eigenvectors. But, what about when they don't? How far away are these eigenvectors?









Turning a fun exploration to practical application

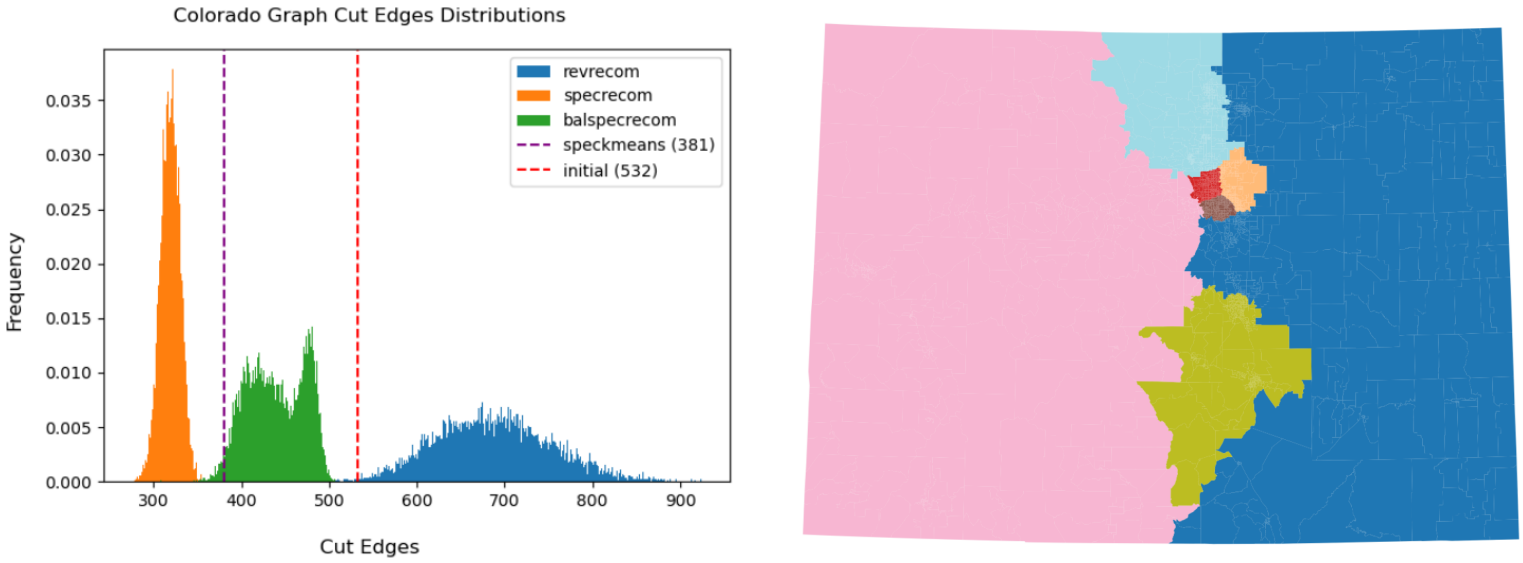


Figure: Using Laplacian eigenvectors, its been shown we can generate compact, connected districted maps [1]. Working with Phousawanh Peaungvongpakdy, we want to analyze convergence properties of these types of algorithms.

Spectral relationship between A and L

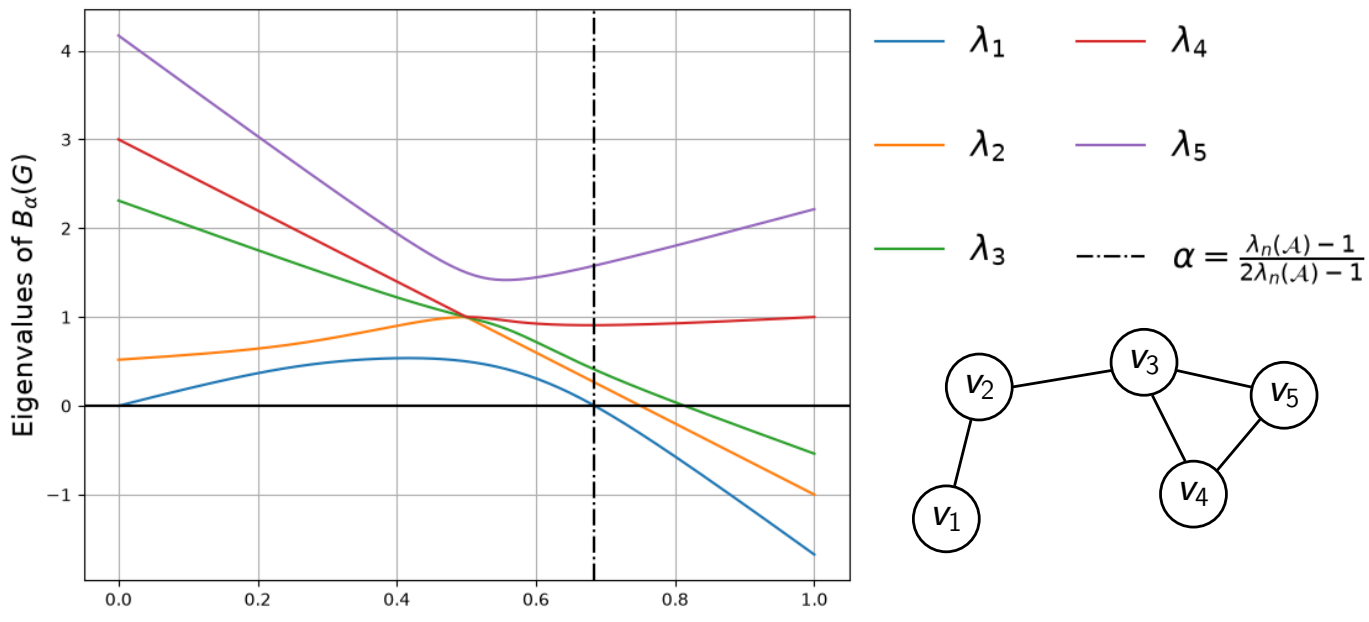


Figure: We have classified B_α transition from positive semi-definite to not exploring the relationship between A and L's spectral properties. What other properties can we characterize? Please let them be something cool about eigenvectors.

Questions?

Thanks!



Figure: On left, Beetle. On right, QR code link to "*Some Spectral Properties of the B_α Matrix of a graph*" on Arxiv.

References



E. Davies, R. Job, M. Kampbell, H. Kim, and H. Seo.
Spectral partitioning of graphs into compact, connected regions, 2025.