# Graphical Inginuety: Spectral Solutions to Combinatorial Conondrums

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# Preface

# SGT Talk Thursday 4/25

## Reimagining Spectral Graph Theory

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**April 25, 2024 at 4:20**
Spark 335

Zoom ID: 992 1304 8021 Passcode: 483214

Refreshments at 3:30pm
Neill 216 (Hacker Lounge)

# Background

# Basics

## Definition

The adjacency matrix of a graph $G$, $A(G) = A = [a_{ij}]$, is the $n \times n$ matrix such that

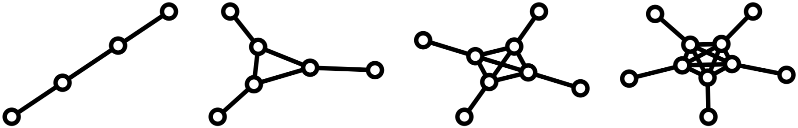
$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ adjacent to } j \text{ in } G \\ 0 & \text{otherwise} \end{cases}$$

for $i, j = 1, 2, \cdots, n$

## Definition

Throughout, the spectrum of a graph $G$ will refer to the collection of eigenvalues of $A$, $\sigma(A)$, with the ordering $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

# Examples I



Adjacency spectra starting from the left where $\phi = \frac{1+\sqrt{5}}{2}$:

- $\sigma(A) = \{-\phi, -\frac{1}{\phi}, \frac{1}{\phi}, \phi\}$
- $\sigma(A) = \{-\phi, -\phi, 1-\sqrt{2}, \frac{1}{\phi}, \frac{1}{\phi} 1+\sqrt{2}\}$
- $\sigma(A) = \{-\phi, -\phi, -\phi, -.302\cdots, \frac{1}{\phi}, \frac{1}{\phi}, \frac{1}{\phi}, 3.302\cdots\}$
- $\sigma(A) = \{-\phi, -\phi, -\phi, -\phi, .236\cdots, \frac{1}{\phi}, \frac{1}{\phi}, \frac{1}{\phi}, \frac{1}{\phi}, 4.236\cdots\}$

# Examples II

- Let $[n] = \{1, 2, \cdots, n\}$.
- Let nodes of graph $G$ correspond to distinct $k$-element subsets of $[n]$.
- If $|i \cap j| = 0$ for $i, j \in V(G)$, then let $i$ be adjacent to $j$ i.e. $(i, j) \in E(G)$.

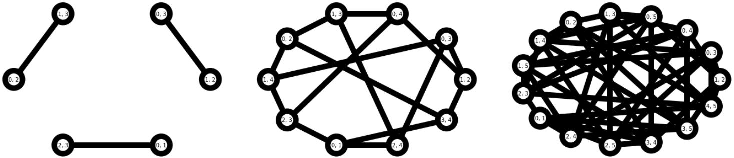


Figure: The Kneser Graphs $K(n, k)$ with $\sigma(A) = \{(-1)^j \binom{n-k-j}{k-j} \mid j = 0, 1, 2, \cdots, k\}$

# Structure and Spectra

# Basic Examples

Let $G$ be a simple, undirected graph on $n$ nodes. Then,

- $\sum_{i=1}^{n} \lambda_i = 0$
- $\mathrm{tr}(A^2) = \sum_{i=1}^{n} \lambda_i^2 = 2m$ where $m$ is the number of edges in the graph
- $\Delta(G) \geq \lambda_1 \geq \bar{d}(G)$ where $\bar{d}(G)$ is the average degree in $G$ and $\Delta(G)$ is the max. Also, $\lambda_1 = \bar{d}(G)$ for $k$-regular graphs where $\bar{d} = k$.

# Hoffman's Bound

**Theorem**

*Let $G = (V, E)$ be finite, $k$-regular graph on $n$ nodes with adjacency matrix $A$. Let $S \subset V(G)$ be an independent set of vertices in $G$. Then*

$$\frac{|S|}{n} \leq \frac{-\lambda_n}{k - \lambda_n}$$

# Proof

- Let $\mathbb{R}^n$ be equipped with inner product

$$\langle u, v\rangle = \frac{1}{n}\sum_{x\in V(G)} u(x)v(x)$$

- Let $S \subset V(G)$ be an independent set with indicator vector $1_S$
- Using the orthonormal basis of eigenvectors for $A$, $\{v_1, v_2, \cdots, v_n\}$, we can rewrite $1_S$ as: $1_S = \sum_{i=1}^n \alpha_i v_i$.
- Let $\mathbf{j}$ be the all ones vector. Note that $\alpha_1 = \langle 1_S, \mathbf{j}\rangle = \frac{|S|}{n} = \langle 1_S, 1_S\rangle$
- Since all $v_i$ are normalized, $\langle 1_S, 1_S\rangle = \sum_{i=1}^n \alpha_i^2$

# Proof

- Lastly, since $0 = \langle A1_S, 1_S \rangle = \sum_{i=1}^{n} \lambda_i \alpha_i^2 \geq d\alpha_1^2 + \lambda_n \sum_{i=2}^{n} \alpha_i^2 = d\alpha_1^2 + \lambda_n(\alpha_1 - \alpha_1^2)$
- Thus, $\frac{|S|}{n} = \alpha_1 \leq \frac{-\lambda_n}{d-\lambda_n}$

# Problem I

# Erdös-Ko-Rado

## Definition

Let $\mathcal{F}$ be a family of sets. If $|X \cap Y| \neq 0 \ \forall \ X, Y \subset \mathcal{F}$, then $\mathcal{F}$ is an *intersecting family*.

## Question

For $n, k \in \mathbb{N}$, what is the maximum size of an intersecting family of $k$-element subsets of an $n$-element set?

# Proof

- Consider the Kneser Graph $K(n,k)$.
- Note that the Kneser Graph is $\binom{n-k}{k}$-regular with spectrum $\{(-1)^j\binom{n-k-j}{k-j} \mid j=0,1,2,\cdots,k\}$ making $\lambda_n = -\binom{n-k-1}{k-1}$.
- So, using Hoffman's Bound,

$$\frac{|S|}{n} \leq \frac{-\lambda_n}{k-\lambda_n} \longrightarrow \frac{\alpha}{\binom{n-k}{k}} \leq \frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k}+\binom{n-k-1}{k-1}}$$

$$\iff \alpha \leq n\frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k}+\binom{n-k-1}{k-1}} = \binom{n-1}{k-1}$$

# Problem II

# Dimer Problem

- The 'dimer' problem named after the two dimensional lattices of 'diatomic' molecules that form on surface of crystals.
- Dimers are forced to occupy two neighboring points on the lattice.

**Question**

How many ways can dimers cover the surface of the crystal?

**Question**

How many perfect matchings, $p$, are there for an $m \times n$ grid?

Short answer: take the permanent! $(\text{per}(A) = p^2)$

## Count

- Let the undirected graph $G$ be an $m \times n$ grid with adjacency matrix $A$ and $H$ be the graph obtained by replacing each edge in $G$ with oppositely oriented edges.
- Note $\text{perm}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)} = p^2 = \sum_{L \subset H} 1$ where $L$ is a spanning subgraph of $H$.
- Likewise, $\det(A) = \sum_{\sigma \in S_n} \left(\text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}\right) = \sum_{L \subset H} (-1)^{c(L)}$ where $c(L)$ is the number of components in $L$.
- For every $L \subset H$, $(-1)^{c(L)} = i^{h(L)}$ where $h(L)$ is the number of horizontal edges in $L$.
- $A$ can be factored as $A_m \otimes \mathbb{I}_n + A_n \otimes \mathbb{I}_m$.

# Count

- Take $A* = A_m \otimes \mathbb{I}_n + iA_n \otimes \mathbb{I}_m$ (multiplying each horizontal edge by $i$.
- Then, since $(-1)^{c(L)} = i^{h(L)}$, $\text{per}(A) = \det(A*)$ and so $\det(A*) = p^2$
- Thus, $p^2 = \prod_{i=1}^{n} \lambda_i$ for $\lambda_i \in \sigma(A*)$ i.e.

$$p^2 = \prod_{j=1}^{m} \prod_{k=1}^{n} \left( 2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1} l\right)\right)$$

# Problem 3: Open

## Bounded Permutations

- Consider permutations $\pi$ of $[n]$ where $|\pi(i) - i| \leq k$ for $i = 1, 2, \cdots, n$
- Denote the number of restrictions of this type by $N(\mathbf{R}^{(k)}, n)$
- Let $k < h < n$ with $k << n$ and consider a partially constructed permutation of the type $\mathbf{R}^{(k)}$:

$$\begin{bmatrix} 1 & 2 & \cdots & h-1 \\ \pi(1) & \pi(2) & \cdots & \pi(h-1) \end{bmatrix}$$

- The next step in our permutation is to select a value for $\pi(h)$
- At this point there are a certain number of 'states of incompletion' of the permutation
- Assign each node $i$ in a digraph $G$ a distinct state such that $i$ is adjacent to $j$ if and only if there is a value of $\pi(h)$ which will take state $i$ and product state $j$.

# Bounded Permutations

## Conjecture

*The limit*

$$\lim_{n\to\infty} N(\mathbf{R}^{(k)}, n)^{1/n}$$

*exists and is equal to the largest eigenvalue of $G$.*

## Question

What about for permutations $\pi$ such that $j \leq \pi(i) - i \leq k$?