# A Look at Limits in Random Graphs

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# The Prairie

## Definition

If $x_n = \{x_1, x_2, x_3, x_4, \dots\}$ is a sequence, then

$$\lim_{n\to\infty} x_n = x$$

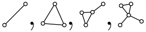
if for every $\epsilon > 0$ there exists some $N > 0$ such that

$$|x_n - x| < \epsilon \quad \text{when} \quad n \geq N \quad \text{for every natural number } n$$

i.e. under certain conditions your sequence will get arbitrarily *close* to some limiting object

## The Prairie

### Definition (?)

If $G_n = \{$  $\}$ is a sequence, then

$$\lim_{n\to\infty} G_n =$$ 

if for every $\epsilon > 0$ there exists some $N > 0$ such that

$$|G_n -$$  $$| < \epsilon \quad \text{when} \quad n \geq N \quad \text{for every natural number } n$$

i.e. What do we want arbitrarily *close* to mean in the space of graphs? What conditions would we then require for a limit to exist?

## The Treeline

### Theorem

*If $f_n = \{f_1, f_2, f_3, f_4, \ldots\}$ is a sequence of continuous functions such that $f_n$ converges uniformly to $f$, then $f$ is continuous.*

i.e. under certain conditions your sequence will get arbitrarily *close* to some limiting object *and* you know something about the object

# The Treeline

## Theorem (?)

*If* $f_n(G_n) = \{f_1($$), f_1($$), f_2($$), f_3($$), \dots\}$ *is a sequence of ________ functions such that* $f_n(G_n)$ *converges ________ to* $f($$)$*, then* $f($$)$ *is ________.*

i.e. After we have a notion of arbitrary *closeness*, what properties can we infer about limiting objects? What restrictions do we need for the convergence if any?

# Pointed vs. Rooted

## Definition

A *pointed* graph is a pair $g^\bullet = (g, \rho)$ where $\rho \in V(g)$ is a distinguished node. Let $\mathcal{G}^\bullet$ denote the set of (locally finite, connected, countable) pointed graphs up to isomorphism.

## Definition

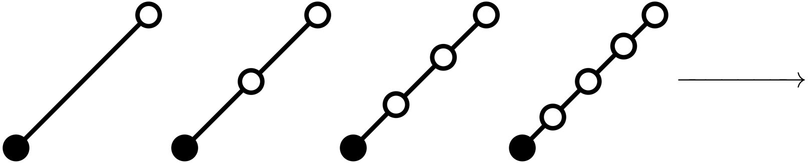
A *rooted* graph is a pair $\vec{g} = (g, \vec{e})$ where $\vec{g} \in E(g)$ is a distinguished edge. Let $\vec{\mathcal{G}}$ denote the set of (locally finite, connected, countable) rooted graphs up to isomorphism.

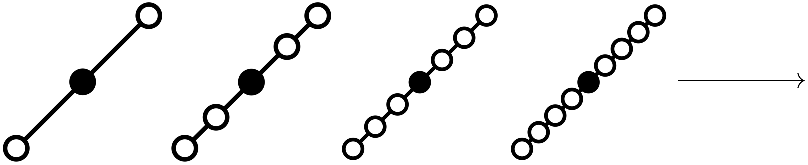
Defining sequences that converge considering the *global* geometry of the graphs is relatively straight forward and boring. We care about *local* geometry of graphs i.e. subgraphs around a root node $\rho$, $B_r(G, \rho)$. Allowing us to define distances:

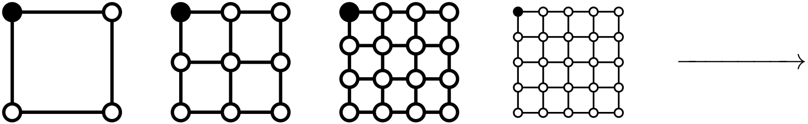
- Are the building blocks of radius $r$ in two graphs different? $\mathbb{1}_{B_r(G,\rho)\not\cong B_r(G',\rho')}$
- If so, they contribute $2^{-r}$ to the distance.
- Look at all blocks of radius $r$:

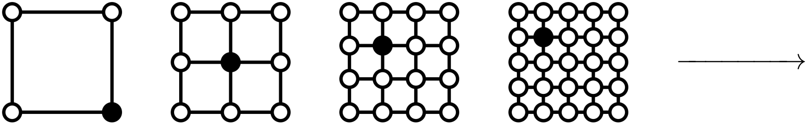
$$d_{loc}(G, G') = \sum_{r\geq 0} \min(1, \mathbb{1}_{B_r(G,\rho)\not\cong B_r(G',\rho')})2^{-r}$$

So, a sequence of graphs $G_n$ converges if the local structure around the root stabilizes. What it stabilizes to determines the limiting object.









# Equivalent Definition of Convergence

- So, we have a way of defining convergence locally.
- Using the local topology, we can take a fixed graph sequence and find its limit as $n$ grows.
- But, we'd like to do the same with random graph sequences too!

## Definition

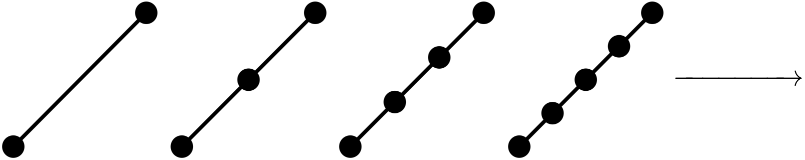
Let $g$ be some finite graph. The *uniformly pointed version* of $g$ is the random graph, $\mathrm{U}^{\bullet}(g)$, whose law is $\frac{1}{|V|}\sum_{x\in V(g)}\delta_{(g,x)}$

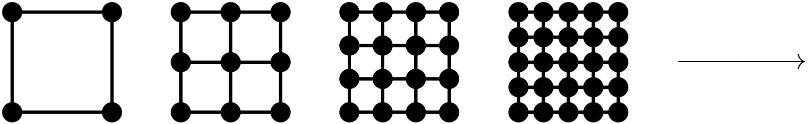
Equivalently,

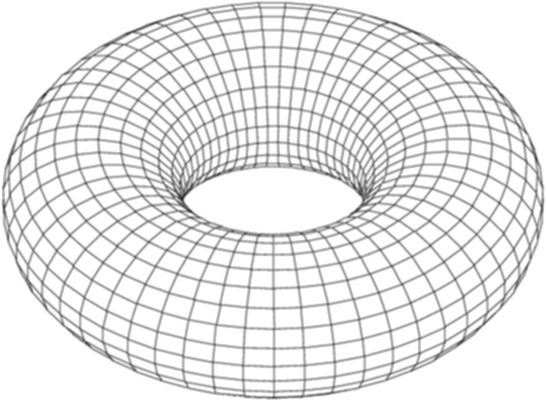
$$\mathbb{E}[f(\mathrm{U}^{\bullet}(g))] = \frac{1}{|V|}\sum_{x\in V(g)} f(g,x)$$

(for any measurable $f : \mathcal{G}^{\bullet} \to \mathbb{R}$)

That is, $\mathrm{U}^{\bullet}(g)$ is the random graph obtained by considering neighborhoods of uniformly chosen nodes. In other words, expressing what the "average" local geometry behaves like







## Benjamini-Schramm Convergence

### Definition

A sequence of unpointed random finite graphs $G_n$ converges in the Benjamini-Schramm sense towards $G_\infty^\bullet = (G_\infty, \rho_\infty)$ if under $d_{loc}$, $\mathrm{U}^\bullet(G_n)$ converges in distribution to $G_\infty^\bullet$.
Equivalently if for any bounded continuous $\phi : \mathcal{G}^\bullet \to \mathbb{R}_+$,

$$\mathbb{E}\left[\frac{1}{|V|}\sum_{x\in V(G_n)} \phi(G_n, x)\right] \to \mathbb{E}[\phi(G_\infty^\bullet)]$$

1. Convergence
2. Local Topology
3. **Pointed Graphs**
4. Interlude
5. Rooted Graphs
6. Main Idea (and yes, SGT)

# Unimodularity

## Definition

A random pointed graph $(G, \rho)$ is *unimodular* if

$$\mathbb{E}\left[\sum_{x \in V(G)} f(G, \rho, x)\right] = \mathbb{E}\left[\sum_{x \in V(G)} f(G, x, \rho)\right]$$

(for any $f : \mathcal{G}^{\bullet\bullet} \to \mathbb{R}_+$)

That is, the average mass a distinguished node receives equals the average mass it sends.

## Stability

### Theorem

*Let $G_n$ be a sequence of graphs that converges in the Benjamini-Schramm sense. If $U^{\bullet}(G_n)$ is unimodular and converges to $G_{\infty}^{\bullet}$, then $G_{\infty}^{\bullet}$ is also unimodular.*

# Why is unimodularity nice?

We can pick $f(G, \rho, x)$!









1. Convergence
2. Local Topology
3. Pointed Graphs
4. Interlude
5. **Rooted Graphs**
6. Main Idea (and yes, SGT)

# Defining the SRW

Simple random walk:

- Take a graph $g$.
- Pick a vertex $x \in V(g)$ to start at.
- Looking at the neighbors of $x$ in $g$, pick one uniformly at random.
- Move to that vertex.
- Repeat previous two steps.

# Stationarity and Reversibility

## Definition

Let $\vec{G} = (\vec{G}, \vec{e})$ be a random rooted graph. Let $\{\vec{E}_i\}_{i \geq 0}$ denote the sequence of visited edges during a simple random walk on $\vec{G}$.

- If the law of $(\vec{G}, \vec{E}_k)$ is the same as $(\vec{G}, \vec{e})$ for every $k \geq 0$, $\vec{G}$ is *stationary*.
- If the law of $(\vec{G}, \vec{E}_0)$ is the same as $(\vec{G}, \overleftarrow{E}_0)$, $\vec{G}$ is *reversible*.

## Stability

### Theorem

*Let $G_n$ be a sequence of stationary and reversible random graphs which converges to $\vec{G}_\infty$ in the Benjamini-Schramm sense. Then $\vec{G}_\infty$ is stationary and reversible.*

## Correspondence!

### Theorem

*Let $G^\bullet = (G, \rho)$ be a pointed unimodular random graph. Let the rooted graph $\vec{G} = (G, \rho_e)$ denote the graph obtained by choosing an edge on $\rho$ uniformly at random. Biasing by the degree of $\rho_e$ returns a stationary and reversible random graph.*

### Theorem

*Let $\vec{G} = (G, \vec{e})$ be a rooted stationary and reversible random graph. Let $G^\bullet = (G, \vec{e}_*)$ denote the pointed graph obtained by distinguishing the origin node of $\vec{e}$, $\vec{e}_*$. Biasing by the inverse degree of $\vec{e}_*$ returns a unimodular random graph.*

## The Framework

A major portion of this body of work can be summarized as follows:

- Find a Benjamini-Schramm limit for a sequence of graphs and find its law.
- Pick your favorite graph parameter.
- Determine whether we can use the law of the limiting object to find the limit of the graph parameter on our sequence i.e. for a graph parameter $\phi(G_n)$, $\mathbb{E}[\mathrm{U}^{\bullet}(G_n)] \to \mathbb{E}[G_{\infty}^{\bullet}] \implies$ the limit of $\phi(G_n)$ exists and can be found using the law of $G_{\infty}^{\bullet}$

## Participating Properties

Among the graph parameters $\phi$ that we know this is true are:

- number of spanning trees
- size of maximum matching
- rank of adjacency matrix
- maximal subgraph density
- spectral statistics!

## Spectral Convergence

### Theorem

*If $G_n$ converges to $G^{\bullet}_{\infty}$ in the Benjamini-Schramm sense where $\mu_{\mathcal{G}_\backslash}$ and $\mu_{\mathcal{G}_\infty}$ denote the corresponding laws, then*

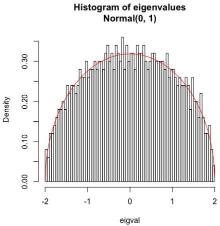
$$\sup_{\lambda \in \mathbb{R}} |\mu_{G_n}((-\infty, \lambda]) - \mu_{G_\infty}((-\infty, \lambda])| \to 0$$

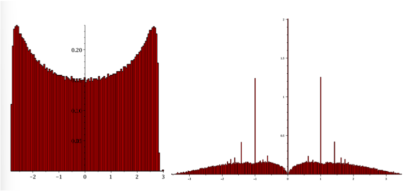
*as $n \to \infty$*

In other words, we can know that the spectral distribution concentrates and (oftentimes) what it concentrates to!

# Utility

This is particularly helpful since theorems like Wegner's semi-circle law only apply to dense graphs. With the help of Benjamini-Schramm, we can know more about larger regimes of graphs including (most importantly) sparse ones.



# End

- Thanks!
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# References


- "Spectra of sparse random graphs" by Justin Salez link: here.
- "Eigenvalues of random matrices..." by Kenneth Tay link: here.
- "A random walk among random graphs" Nicolas Curien link: here.